University of Mumbai
F.Y.B.Sc. (Mathematics)
Practical Manual
w.e.f. 2020-21

Contents

Preface

It has been our academic endeavour to cater to the needs of the students since the introduction of credit based system by the University of Mumbai where the focus has been shifted from teacher centric to learner-centric.
The two papers of F.Y.B.Sc Mathematics as per the new syllabi are:

	Semester I	Semester II
Paper I	Calculus I	Calculus II
Paper II	Algebra I	Discrete Mathematics

Keeping in mind the need to develop among students an ability to understand and apply theoretical concepts, in papers I and II practicals have been prescribed.

In a meeting of the Board of Studies in Mathematics held along with the members of the syllabus restructuring committee, it was decided to frame the Practical Question sets as per the guidelines given in the revised syllabus in each paper of Mathematics of F.Y./S.Y./T.Y. B.Sc./B. A. This is in tune with a circular in which the University Grants Commission had recommended that the Board of Studies in each subject should prepare a Question bank for each course.
The following committee was formed and entrusted with this collective responsibility for F.Y.B.Sc/B.A.

Name	College
Mrs. Anuradha Namjoshi (Chairperson)	**Sathaye College**
Prof. Sangeeta Joshi	Smt C.H.M. College ,Ulhasnagar
Prof. Urmila Pillai	Smt C.H.M. College ,Ulhasnagar
Dr. Rajesh Raut	RD & SH National college and S WA Science college.
Prof. Amit Gawde	Patkar and Varde College.
Prof. Salil Savarkar	Smt C.H.M. College ,Ulhasnagar
Prof. Komal Wategaonkar	Royal College of Arts, science and commerce.
Prof. Bhakti Velankar	NES Ratnam college.
Prof. Pooja Rochani	Smt C.H.M. College, Ulhasnagar.
Prof. Pramod Tohake	MD College.

The committee focused on the following aspects:

(1) Uniform notations and definitions in the list of reference books in the syllabus.

(2) Collating relevant questions for the revised syllabus from recommended books.

0.0

(3) Preparing a comprehensive practical question sets consisting of objective and descriptive questions which streamlines and structures the syllabi.

(4) Providing a ready reference for teachers and students alike, to pinpoint the highlights of the revised syllabi.

(5) The practical on miscellaneous theory questions consists of core theorems and problems based on them have been compiled from all reference books in the syllabi.

The aim is to achieve the above objectives during the practical sessions.

These suggested practicals is a combined effort of the committee members and teachers of the colleges affiliated to University of Mumbai. Numerous exercises are given under each section for the students to practice and test his/her comprehension and ability. While many of these problems may be covered during practical, we expect the student to work out the remaining ones. We hope these practical sets provide students with a good grounding in the fundamentals in each topic.

We thank Prof Deore,Chairman of B.O.S. in mathematics and Dr. Santosh Shende Convenor of the Syllabus Committee for making valuable suggestions and discussing important aspects of the syllabi with the committee members.

We also thank Prof. Veena Bhakta Kamat for coordinating and organizing the above activity that shaped the manual as it is today.

We appreciate inputs given by Prof. Sunil Chokhani and Dr. Manisha Acharya which set this work possible as it is till the end.

We are extremely grateful to Prof. Balmohan Limaye, Prof. Sudhir Ghorpade and Prof. Vinayak Kulkarni for their valuable insights, suggestions and comments.

A special mention to Dr. Abhaya Chitre, Member of B.O.S. in Mathematics for accomplishing this herculean task of compiling and putting this together in the present form.

Finally, we take full responsibility for any error that may have inadvertently crept into this hand book and would appreciate if any such errors are brought to our notice.

....

SCHEME OF EVALUATION

The performance of the student is evaluated as follows:

(I) **Internal Evaluation of 25 marks:**
F.Y.B.Sc.

(i) One Class Test of 20 marks to be conducted during Practical session.
Paper pattern of the Test:
Q1: Definitions / Fill in the blanks / True of False with Justification (04 Marks).
Q2: Multiple choice 5 questions (10 marks: 5×2).
Q3: Attempt any 2 from 3 descriptive questions (06 Marks: 2×3).

(ii) Active participation in routine class: 05 Marks.

F.Y.B.A.

(i) One Class Test of 20 marks to be conducted during Tutorial session.
Paper pattern of the Test:
Q1: Definitions / Fill in the blanks / True of False with Justification (04 Marks).
Q2: Multiple choice 5 questions (10 marks: 5×2).
Q3: Attempt any 2 from 3 descriptive questions (06 Marks: 2×3).

(ii) Journal: 05 Marks.

(II) **Semester End Theory Examinations:** There will be a Semester-end external Theory examination of 75 marks for each of the course USMT101/UAMT101, USMT102 of Semester I and USMT201/UAMT201, USMT202 of Semester II to be conducted by the college.

1. Duration: The examinations shall be of 2 and $\frac{1}{2}$ hours duration.
2. Theory Question Paper Pattern:
   a) There shall be FOUR questions. The first three questions Q1, Q2, Q3 shall be of 20 marks, each based on the units I, II, III respectively. The question Q4 shall be of 15 marks based on the entire syllabus.
   b) All the questions shall be compulsory. The questions Q1, Q2, Q3, Q4 shall have internal choices within the questions. Including the choices, the marks for each question shall be 25-27.
   c) The questions Q1, Q2, Q3, Q4 may be subdivided into sub-questions as a, b, c, d & e, etc and the allocation of marks depends on the weightage of the topic.

(III) **Semester End Examinations Practicals:**
At the end of the Semesters I & II Practical examinations of three hours duration and 100 marks shall be conducted for the courses USMTP01, USMTP02.
In semester I, the Practical examinations for USMT101 and USMT102 are held together by the college.
In Semester II, the Practical examinations for USMT201 and USMT202 are held together by the college.

Paper pattern: The question paper shall have two parts A and B.
Each part shall have two Sections.

Section I Objective in nature: Attempt any Eight out of Twelve multiple choice questions (04 objective questions from each unit) ($8 \times 3 = 24$ Marks).

Section II Problems: Attempt any Two out of Three (01 descriptive question from each unit) ($8 \times 2 = 16$ Marks).

Practical Course	Part A	Part B	Marks out of	duration
USMTP01	Questions from USMT101	Questions from USMT102	80	3 hours
USMTP02	Questions from USMT201	Questions from USMT202	80	3 hours

Marks for Journals and Viva:
For each course USMT101/UAMT101, USMT102, USMT201:

1. Journal: 10 marks (5 marks for each journal).
2. Viva: 10 marks.

Each Practical of every course of Semester I and II shall contain at least 10 objective questions and at least 6 descriptive questions.
A student must have a certified journal before appearing for the practical examination.
In case a student does not posses a certified journal he/she will be evaluated for 80 marks. He/she is not qualified for Journal + Viva marks.

xxxxxxxx

Reference Books:

1. R. R. Goldberg, Methods of Real Analysis, Oxford and IBH, 1964.
2. K. G. Binmore, Mathematical Analysis, Cambridge University Press, 1982.
3. R. G. Bartle- D. R. Sherbert, Introduction to Real Analysis, John Wiley & Sons, 1994.
4. Sudhir Ghorpade and Balmohan Limaye, A course in Calculus and Real Analysis, Springer International Ltd, 2000 (second edition). More information about this book is available at: http://www.math.iitb.ac.in/ srg/acicara/
5. G. F. Simmons, Differential Equations with Applications and Historical Notes, McGraw Hill, 1972.
6. E. A. Coddington , An Introduction to Ordinary Differential Equations.Prentice Hall, 1961.
7. W. E. Boyce, R. C. DiPrima, Elementary Differential Equations and Boundary Value Problems, Wiely, 2013.

Additional Reference Books

1. T. M. Apostol, Calculus Volume I, Wiley & Sons (Asia) Pte, Ltd.
2. Richard Courant-Fritz John, A Introduction to Calculus and Analysis, Volume I, Springer.
3. Ajit kumar and S. Kumaresan, A Basic Course in Real Analysis, CRC Press, 2014.
4. James Stewart, Calculus, Third Edition, Brooks/ cole Publishing Company, 1994.
5. D. A. Murray, Introductory Course in Differential Equations, Longmans, Green and Co., 1897.
6. A. R. Forsyth, A Treatise on Differential Equations, MacMillan and Co.,1956.

Chapter 1

(USMT 101) Calculus I

1.1 Practical 1.1: Algebraic and Order Properties of Real Numbers and Inequalities

1.1.1 Prerequisite of Practical 1.1

(1) **Axioms for addition**

(i) **Closure property:** $\mathbb{R}$ is closed under addition.
That is, $x, y \in \mathbb{R} \Longrightarrow x + y \in \mathbb{R}$.

(ii) **Addition is associative:** $x + (y + z) = (x + y) + z, \quad$ for all $\; x, y, z \in \mathbb{R}$.

(iii) **Addition is commutative:** $x + y = y + x, \quad$ for all $\; x, y \in \mathbb{R}$.

(iv) **Existence of an additive identity:**
There exists an element 0 in $\mathbb{R}$ such that $x + 0 = x, \quad$ for all $\; x \in \mathbb{R}$.

(v) **Existence of an additive inverse:**
For each $x \in \mathbb{R}$ there exists x' in $\mathbb{R}$ such that $x + x' = 0$.

(2) **Axioms for multiplication**

(i) **Closure property:** $\mathbb{R}$ is closed under multiplication.
That is, $x, y \in \mathbb{R} \Longrightarrow xy \in \mathbb{R}$.

(ii) **Multiplication is associative:** $x(yz) = (xy)z, \quad$ for all $\; x, y, z \in \mathbb{R}$.

(iii) **Multiplication is commutative:** $xy = yx, \quad$ for all $\; x, y \in \mathbb{R}$.

(iv) **Existence of a multiplicative identity:**
There exists an element 1 in $\mathbb{R}$ such that $x \cdot 1 = x, \quad$ for all $\; x \in \mathbb{R}$.

(v) **Existence of multiplicative inverse:**
For each $x \in \mathbb{R}$, $x \neq 0$, there exists an element x^* in $\mathbb{R}$ such that $xx^* = 1$.

(3) **Axioms about distributivity of multiplication over addition** Multiplication is **distributive** over addition:

$$x(y + z) = xy + xz \; \forall \; x, y, z \in \mathbb{R}.$$

(4) **Elementary Results**

(i) Additive identity is unique.
(ii) Multiplicative identity is unique.
(iii) Every real number has unique additive inverse. (The unique additive inverse of x is denoted by $-x$.)
(iv) Every nonzero real number has unique multiplicative inverse. (The unique multiplicative inverse of $x \neq 0$ is denoted by x^{-1} or by $\frac{1}{x}$)
(v) (i) The unique additive inverse of x is denoted by $-x$.
(ii) The unique multiplicative inverse of non-zero x is denoted by x^{-1} or by $\frac{1}{x}$.
(vi) Let $a, b \in \mathbb{R}$. Then
(i) there is exactly one $x \in \mathbb{R}$ such that $a + x = b$.
(ii) if $a \neq 0$, there is exactly one $x \in \mathbb{R}$ such that $ax = b$.

(5) If $a, b, c \in \mathbb{R}$, then using only the algebraic properties, we can prove the following.

(i) $-(-a) = a$
(ii) $(a^{-1})^{-1} = a$
(iii) $a0 = 0 = 0a$
(iv) $(-a)b = -(ab) = a(-b)$
(v) $(-a)(-b) = ab$
(vi) $a(b - c) = ab - ac$
(vii) If $ab = 0$ then $a = 0$ or $b = 0$
(viii) If $a \neq 0, b \neq 0$, then $(ab)^{-1} = b^{-1}a^{-1}$
(ix) If $a \neq 0$, then $(-a)^{-1} = -a^{-1}$
(x) $-0 = 0$
(xi) $1^{-1} = 1$

(6) **Order properties of $\mathbb{R}$:**
We shall assume that there exists a non-empty subset $\mathbb{R}^+ \subset \mathbb{R}$, called the set of positive real numbers, which satisfies the following properties:

(I) If $x, y \in \mathbb{R}^+$ then $x + y \in \mathbb{R}^+$.
(II) If $x, y \in \mathbb{R}^+$ then $xy \in \mathbb{R}^+$.
(III) **Trichotomy Law**: For every $x \in \mathbb{R}$, exactly one of the following is true:
$x = 0$ or $x \in \mathbb{R}^+$ or $-x \in \mathbb{R}^+$.

(Note: $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$.)

(7) **Order relation on $\mathbb{R}$:**
For $x, y \in \mathbb{R}$, define x to be **less than** y, and write $x < y$, if $y - x \in \mathbb{R}^+$.
Sometimes, we write $y > x$ in place of $x < y$ and say that y is **greater than** x.

Definition of symbols $\leq$ and $\geq$ called **less than or equal to** and **greater than or equal to** respectively, as follows:
$x \leq y$ if either $x < y$ or $x = y$.
$y \geq x$ if $x \leq y$.
(Note: Law of Trichotomy can also be stated as follows:
For $x, y \in \mathbb{R}$, exactly one of the three relations holds : $x = y$, $x < y$, $y < x$).

(8) **Elementary Results of order relation on $\mathbb{R}$:**
If x, y, z are real numbers, then

(i) $x < 0 \implies -x > 0$

(ii) $x < y$ and $y < z \Longrightarrow x < z$

(iii) $x < y \Longrightarrow x + z < y + z$

(iv) $x < y$ and $z > 0 \Longrightarrow xz < yz$

(v) $x < y$ and $z < 0 \Longrightarrow xz > yz$

(vi) $x > 0, y < 0 \Longrightarrow xy < 0$

(vii) $1 > 0$.

(viii) If $x > 0$, then $x^{-1} > 0$ and if $x < 0$ then $x^{-1} < 0$.

(ix) If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.

(x) If $x < y < 0$, then $y^{-1} < x^{-1} < 0$.

(9) Given any $a \in \mathbb{R}$ and $n \in \mathbb{N}$, the n^{th} **power** of a (denoted by a^n) is defined to be the product $a \cdots a$ of a with itself taken n times.
If $a \neq 0$, then $a^0 = 1$ and $a^{-n} = \left(\frac{1}{a}\right)^n$.
Thus integral powers of all nonzero real numbers are defined.
Following properties can be proved using the algebraic properties and the order properties of $\mathbb{R}$.

(i) $(xy)^n = x^n y^n$ for all $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$ (with $xy \neq 0$ if $n \leq 0$).

(ii) $(x^m)^n = x^{mn}$ and $x^{m+n} = x^m x^n$ for all $m, n \in \mathbb{Z}$ and $x \in \mathbb{R}$ (with $x \neq 0$ if $m \leq 0$ or $n \leq 0$).

(iii) If $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$ with $0 \leq x < y$, then $x^n < y^n$.

The first two properties above are called **Laws of exponents** or the **Laws of indices** for integral powers.

(10) (The proof of the following statement is not to be expected from the students)
Given any $n \in \mathbb{N}$ and $a \in \mathbb{R}$ with $a \geq 0$, there exists a unique $b \in \mathbb{R}$ such that $b \geq 0$ and $b^n = a$.
Using this theorem, root of a non-negative real number is defined.
For $n \in \mathbb{N}$ and $a \in \mathbb{R}$ with $a \geq 0$, the n^{th} **root** of a is defined as the unique real number b such that $b \geq 0$ and $b^n = a$. We denote this real number by $\sqrt[n]{a}$ or by $a^{\frac{1}{n}}$. Thus $a^{\frac{1}{n}} \geq 0$ for all $a \geq 0$.
In case $n = 2$, we write $\sqrt{a}$ instead of $\sqrt[2]{a}$.
The following properties of the n^{th} roots can be proved.

(i) $(xy)^{\frac{1}{n}} = x^{\frac{1}{n}} y^{\frac{1}{n}}$ for all $n \in \mathbb{Z}$ and $x, y \in \mathbb{R}$ (with $xy \neq 0$ if $n \leq 0$).

(ii) $(x^{\frac{1}{m}})^{\frac{1}{n}} = x^{\frac{1}{mn}}$ and $x^{\frac{1}{m} + \frac{1}{n}} = x^{\frac{1}{m}} x^{\frac{1}{n}}$ for all $m, n \in \mathbb{Z}$ and $x \in \mathbb{R}$ (with $x \neq 0$ if $m \leq 0$ or $n \leq 0$).

(iii) If $n \in \mathbb{N}$ and $x, y \in \mathbb{R}$ with $0 \leq x < y$, then $x^{\frac{1}{n}} < y^{\frac{1}{n}}$.

(11) Given any $r \in \mathbb{Q}, r = \dfrac{m}{n}$, where $m, n \in \mathbb{Z}$ with $n > 0, a^r$ is defined as $a^r = (a^m)^{\frac{1}{n}}$ for $a \in \mathbb{R}$, with $a > 0$.
Thus rational powers of positive real numbers are defined.

(12) For negative real numbers, **some non-integral rational powers are not defined in** $\mathbb{R}$. For example, $(-1)^{\frac{1}{2}}$ cannot equal any $b \in \mathbb{R}$ as $b^2 \geq 0$.

(13) **If n is odd and $a \in \mathbb{R}$ with $a > 0$** then $(-a)^{\frac{1}{n}} = -(a^{\frac{1}{n}})$.
Hence, for $x \in \mathbb{R}, x \neq 0$, the $r^{\mathbf{th}}$ **power** x^r is defined whenever $r \in \mathbb{Q}, r = \dfrac{m}{n}$ where $m \in \mathbb{Z}, n \in \mathbb{N}$ with n odd.

(14) If r is any positive rational number, then $0^r = 0$ and 0^0 is not defined.

(15) If x is a real number, the **absolute value** of x is a nonnegative real number, denoted by $|x|$, defined as follows:

$$|x| := \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

Geometrically, $|x|$ is the distance of x from zero.

(16) **Properties of absolute value** For $x, y \in \mathbb{R}$, the following properties hold.

(i) $|x| \geq 0$.
(ii) $|x| = 0 \iff x = 0$
(iii) $|x| = |-x|$.
(iv) $|x| = \max\{x, -x\}$.
(v) $-|x| \leq x \leq |x|$.
(vi) $|xy| = |x||y|$.
(vii) If $y \neq 0$ then $\left|\dfrac{1}{y}\right| = \dfrac{1}{|y|}$.
(viii) If $y \neq 0$ then $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$.
(ix) If $r \in \mathbb{R}, r > 0$, then $|x| \leq r$ if and only if $-r \leq x \leq r$.
(x) $|x + y| \leq |x| + |y|$.
(xi) $|x - y| \leq |x| + |y|$.
(xii) $|x - y| \geq \Big||x| - |y|\Big|$.

(17) (i) $|a^n - b^n| \leq (n-1)M^n|a - b|$, where $M = \max\{|a|, |b|\}$.
(ii) $\left|a^{1/n} - b^{1/n}\right| \leq |a - b|^{1/n}$, provided $a \geq 0$ and $b \geq 0$.

(18) For $a, b \in \mathbb{R}$ the **Arithmetic Mean** (A.M.) of a and b is $\dfrac{a+b}{2}$.
For $a, b \in \mathbb{R}^+$, the **Geometric Mean** (G.M.) of a and b is $\sqrt{a * b}$.

If $a, b \in \mathbb{R}^+$ and A.M. and G.M. are their arithmetic and geometric mean respectively then A.M. $\geq$ G.M.
If $a_1, a_2, \cdots, a_n$ are non-negative real numbers, then $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{\frac{1}{n}}$.

(19) **Cauchy-Schwarz Inequality**
If $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are any real numbers then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} {b_k}^2\right).$$

That is,

$$\left(\sum_{k=1}^{n} a_k b_k\right) \leq \left(\sum_{k=1}^{n} {a_k}^2\right)^{\frac{1}{2}}\left(\sum_{k=1}^{n} {b_k}^2\right)^{\frac{1}{2}}.$$

1.1.2 PRACTICAL 1.1

(A) Objective Questions

Choose correct alternative in each of the following:

(A) Choose correct alternative in each of the following:

(1) The multiplicative identity in $\mathbb{R}$ is

(a) 0 (b) 1 (c) -1 (d) 3

(2) The additive identity in $\mathbb{R}$ is

(a) 1 (b) 2 (c) -1 (d) 0

(3) For $x \in \mathbb{R}$, additive inverse of x

(a) Always exists
(b) Exists only if $x \neq 0$.
(c) Exists only if $x > 0$
(d) None of these.

(4) For $x \in \mathbb{R}$, multiplicative inverse of x

(a) Always exists
(b) Exists only if $x \neq 0$
(c) Exists only if $x > 0$.
(d) None of these.

(5) The multiplicative inverse of non-zero real number x is denoted by

(a) x^{-1} (b) $-x$ (c) $-(-x)$ (d) $\sqrt{x}$

(6) The additive inverse of non-zero real number x is denoted by

(a) x^{-1} (b) $-x$ (c) $-(-x)$ (d) $\sqrt{x}$

(7) The existence of the additive identity in $\mathbb{R}$ is expressed using equation

(a) $x * 1 = x$ (b) $x + 0 = x$ (c) $x * 0 = 0$ (d) $x * \frac{1}{x} = 1$

(8) If $0, 0' \in \mathbb{R}$ are such that $x + 0 = 0' + x, \quad \forall x \in \mathbb{R}$ then

(a) $x = 0$
(b) $0 = 0'$
(c) $0 < 0'$
(d) None of the above.

(9) The existence of multiplicative identity in $\mathbb{R}$ is expressed for all $x \neq 0, x \in \mathbb{R}$, using equation

(a) $x * 1 = x$ (b) $x + 0 = x$ (c) $x * 0 = 0$ (d) $x * \frac{1}{x} = 1$

(10) For $x \in \mathbb{R}$, the additive inverse of $-x$ is

(a) does not exist. (b) x (c) $-x$ (d) None of these.

(11) For $x \in \mathbb{R}$, the multiplicative inverse of $\dfrac{x}{2}$

(a) exists only if $x \neq 0$.
(b) does not exist.
(c) always exists in $\mathbb{R}$
(d) None of these.

(12) For $x, y, z \in \mathbb{R}$, if we have $x + y = x + z$ then

(a) $y + z = 0$
(b) $y = z$ only if $x \neq 0$
(c) $y = z$
(d) None of these.

(13) For $x, y, z \in \mathbb{R}$, if we have $x * y = x * z$ then

(a) $y = z$
(b) $y = z$ only if $x \neq 0$
(c) $x = 0$
(d) None of these.

(14) For $x, y, z \in \mathbb{R}$, the distributivity of multiplication over addition is

(a) $(x + y)z = xz + yz$
(b) $(xy) + z = (x + z)(y + z)$
(c) $(x + y)^z = x^z + y^z$
(d) None of these.

(15) For $x, y, z \in \mathbb{R}$, if we have $xy < xz$ then

(a) $y < z$ (b) $y > z$ (c) $y = z$ (d) if $x > 0$ then $y < z$.

(16) For $x, y \in \mathbb{R}$, if we have $0 < x < y$ then

(a) $x^2 < y^2$ (b) $x^2 > y^2$ (c) $xy = 0$ (d) None of these.

(17) For $a, b \in \mathbb{R}$, the equation $x + a = b$ has

(a) unique solution.
(b) no solution.
(c) many solutions
(d) None of these.

(18) For $a, b \in \mathbb{R}$, the equation $x * a = b$ has

(a) infinitely many solutions
(b) may not have a solution.
(c) exactly one solution.
(d) None of these.

(19) For $a, b \in \mathbb{R}$ with $a \neq 0$, the equation $ax = b$ has

(a) infinitely many solutions
(b) may not have a solution.
(c) exactly one solution.
(d) None of these.

(20) For $a, b \in \mathbb{R}$, if $ab = 0$ then

(a) $a = 0$ and $b = 0$
(b) $a + b = 0$
(c) $a = 0$ or $b = 0$
(d) None of these.

(21) The set $\{x \in \mathbb{R} : |x^2 - 8| = 17\}$ is equal to

(a) empty set (b) $\{5\}$ (c) $\{-5, 5\}$ (d) None of these.

(22) Select the statement from below which is true for each $x \in \mathbb{R}$.

(a) $x < x^2$
(b) $|x^2| = |x|^2$
(c) $|x| < \max\{x, -x\}$
(d) $x > 0 \Longrightarrow \frac{1}{x} < 0$.

(23) The statement $|x| = \max\{x, -x\}$ is

(a) true for each $x \in \mathbb{R}$.
(b) false for some $x \in \mathbb{R}$.
(c) false for each positive $x \in \mathbb{R}$.
(d) true only for each negative $x \in \mathbb{R}$.

(24) For $a, b \in \mathbb{R}$, we have $|a + b| = |a| + |b|$ if and only if

(a) $a = \pm b$ (b) $0 < b < a$ (c) $a^2 + b^2 = 0$ (d) $0 \leq ab$..

(25) $*$ Select the statement from below which is true for each $x, y \in \mathbb{R}$

(a) $|x + y| = \Big||x| + |y|\Big|$
(b) $|x + y| \geq \frac{|x| + |y|}{2}$
(c) $\sqrt{|x + y|} \leq \sqrt{|x|} + \sqrt{|y|}$
(d) $|x + y|^2 \leq |x|^2 + |y|^2$.

(26) If $a > 0$ and $x := 4a, y := 6a, z := 9a$, then the geometric mean of x, y, z is

(a) $24a$ (b) $36a$ (c) $216a^3$ (d) $6a$.

(27) If $a > 1$ then the arithmetic mean of a and $\frac{1}{a}$ is

(a) greater than 2.
(b) greater than $\frac{2}{a}$.
(c) greater than $2a$.
(d) equal to 2.

(28) Let a, b be positive real numbers and A, G be the arithmetic mean and the geometric mean of a, b respectively. Then

(a) $G^2 \leq A^2$
(b) $\frac{1}{G^2} \leq \frac{1}{A^2}$
(c) $\frac{A + G}{2} \leq \sqrt{AG}$
(d) $\sqrt{G} \leq \sqrt{A}$.

(29) $*$ Each element of the following set is an irrational number

(a) $\{\sqrt{2}, \pi, e\}$

(b) $\{\sin\left(\frac{n\pi}{4}\right) : n \in \mathbb{N}\}$

(c) $\{(\sqrt{2})^n : n \in \mathbb{N}\}$

(d) $\{\frac{\sqrt{2}}{\sqrt{n}} : n \in \mathbb{N}\}$.

(30) Let a, b be positive rational numbers. Let A, G be the arithmetic mean and the geometric mean of a, b respectively. Then the following pair is a pair of rational numbers.

(a) G and A.

(b) $\dfrac{A+G^2}{2}$ and $\sqrt{4A^2-2G^2}$

(c) $\dfrac{A+G}{2}$ and $\sqrt{AG}$

(d) $\dfrac{A^2+G^2}{2}$ and $\sqrt{A+G}$.

(B) Descriptive Questions

(1) For $x, y, z, w \in \mathbb{R}$, prove the following.

(i) If $x < y$ then $x < \dfrac{x+y}{2} < y$.

(ii) $0 < x < 1 \Longrightarrow 0 < x^2 < x < 1$

(iii) If $0 < x < y$ then $\sqrt{x} < \sqrt{y}$ and $x < \sqrt{xy} < y$.

(iv) If x, y are positive then $x < y \iff x^n < y^n \ \forall\, n \in \mathbb{N}$. (Hint: Use induction method)

(v) $xy < 0 \Longrightarrow (x > 0$ and $y < 0)$ OR $(x < 0$ and $y > 0)$

(vi) $x < y,\ z < w$ then $xw + yz < xz + yw$.

(vii) $|x+y| = |x| + |y|$ if and only if $xy \geq 0$ for all $x, y \in \mathbb{R}$.

(2) Describe the set of all real numbers x satisfying the given inequality.

(i) $x^2 > 3x + 4$.

(ii) $|4x-5| \leq 13$.

(iii) $|x-17| = |2x+8|$

(iv) $|x^2-8| = 17$

(v) $\frac{1}{x} < x$

(vi) $|x-8| = 3\,|x-2|$

(vii) $|x-3| - |x+6| = 5$

(viii) $1 < x^2 < 4$

(3) For $x, y, z \in \mathbb{R}$ show that $x < y < z \iff |x-y| + |y-z| = |x-z|$.

(4) If $a, b, c \in \mathbb{R}^+$, use arithmetic and geometric mean inequality and prove the following statements.

(i) $(a+b)(a+c)(b+c) \geq 8\,abc$. (Hint: First apply AM-GM inequality to a and b then to b and c and then to a and c.)

(ii) $a + \frac{1}{a} \geq 2$. (Hint: apply AM-GM inequality to a and $\frac{1}{a}$.)

(iii) $a^2 + b^2 + c^2 \geq ab + bc + ac$. (Hint: Apply AM-GM inequality to a^2, b^2 then to b^2, c^2 and then to .....

(iv) $\frac{a}{b} + \frac{b}{a} \geq 2$. (Hint: Apply AM-GM inequality to $\dfrac{a}{b}$ and $\dfrac{b}{a}$.)

(v) $a^3 + b^3 + c^3 \geq a^2b + b^2c + c^2a$. (Hint: apply AM-GM inequality to a^3, a^3, b^3 then to b^3, b^3, c^3 and then to c^3, c^3, a^3.)

(vi) $\dfrac{a^2}{b^2}+\dfrac{b^2}{c^2}+\dfrac{c^2}{a^2}\geq \dfrac{b}{a}+\dfrac{b}{c}+\dfrac{c}{a}$. (Hint: apply AM-GM inequality to $\dfrac{a^2}{b^2},\dfrac{b^2}{c^2}$ then to $\dfrac{b^2}{c^2},\dfrac{c^2}{a^2}$ and then to $\dfrac{c^2}{a^2},\dfrac{a^2}{b^2}$.)

(5) Use Cauchy-Schwartz inequality and prove the following statement.
If $a,b,c\in\mathbb{R}$ are positive such that $a+b+c\leq 3$ then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3$. (Hint Apply Cauchy Schwarz Inequality to $\{\sqrt{a},\sqrt{b},\sqrt{c}\}$ and $\left\{\dfrac{1}{\sqrt{a}},\dfrac{1}{\sqrt{b}},\dfrac{1}{\sqrt{c}}\right\}$).

(6) Match the following:

(a_1)	$\lvert x\rvert<3$	(b_1)	$4<x<6$
(a_2)	$\lvert x-1\rvert<3$	(b_2)	$-3<x<3$
(a_3)	$\lvert 3-2x\rvert<1$	(b_3)	$x>3$ or $x<-1$
(a_4)	$\lvert 1+2x\rvert\leq 1$	(b_4)	$x>2$
(a_5)	$\lvert x-1\rvert>2$	(b_5)	$-2<x<4$
(a_6)	$\lvert x+2\rvert\geq 5$	(b_6)	$-\sqrt{3}\leq x\leq -1$ or $1\leq x\leq\sqrt{3}$
(a_7)	$\lvert 5-x^{-1}\rvert<1$	(b_7)	$1<x<2$
(a_8)	$\lvert x-5\rvert<\lvert x+1\rvert$	(b_8)	$x\leq -7$ or $x\geq 3$
(a_9)	$\lvert x^2-2\rvert\leq 1$	(b_9)	$\dfrac{1}{6}<x<\dfrac{1}{4}$
(a_{10})	$x<x^2-12<4x$	(b_{10})	$-1\leq x\leq 0$

(7) Determine whether each of the following is true or false. In each case give a reason for your decision.

(i) $x<5$ implies $|x|<5$

(ii) $|x-5|<2$ implies $3<x<7$

(iii) $|1+3x|\leq 1$ implies $x\geq -\frac{3}{2}$

(iv) There is no real x for which $|x-1|=|x-2|$

(C) Some More Descriptive Questions

(1) For $x,y,z,w\in\mathbb{R}$, prove the following :

(i) $x^2+y^2=0 \iff x=0$ and $y=0$.

(ii) If z,w are positive then $\frac{x}{z}<\frac{y}{w} \iff xw<yz$ and $\frac{x}{z}<\frac{y}{w} \implies \frac{x}{z}<\frac{x+y}{z+w}<\frac{y}{w}$.

(iii) If $0\leq x<y$ then $x^2\leq xy<y^2$.

(iv) $2\,|x|\,|y|\leq x^2+y^2$.

(v) $\sqrt{|x+y|}\leq\sqrt{|x|}+\sqrt{|y|}$ for all $x,y\in\mathbb{R}$.

(2) Solve the following equations, justifying each step by referring to an appropriate property or theorem.

(i) $3x + 2 = 11$

(ii) $x^2 = 3x$

(iii) $2x^2 - 8 = 10$

(iv) $(x-3)(x+1) = 0$

(3) Suppose x, y, a, b, ϵ are real numbers such that $\epsilon > 0$, $|x - a| < \epsilon$ and $|y - b| < \epsilon$. Show that $|xy - ab| < (|a| + |b|)\epsilon + \epsilon^2$.

(4) Suppose $a \in \mathbb{R}$ such that $0 < a < 1$ and $b := 1 - \sqrt{1-a}$. Prove that $0 < b < a$.

(5) Suppose $x, a, \epsilon \in \mathbb{R}$ such that $0 < \frac{a}{2} < x$ and $|x - a| < \epsilon$. Prove that $\left|\frac{1}{x} - \frac{1}{a}\right| \leq \frac{2\epsilon}{a^2}$.

(6) Suppose x, y are positive real numbers. Prove that $x < y$ if and only if $\sqrt{x} < \sqrt{y}$.

(7) Suppose that for each $i \in \{1, 2, \cdots, n\}$, let $x_i, \lambda_i \in \mathbb{R}$ be such that $|x_i| < 1, \lambda_i \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ then show that $\sum_{i=1}^{n} \lambda_i x_i < 1$.

(8) Let $n \in \mathbb{N}$ and each $r_1, r_2, \cdots, r_n \in \mathbb{R}$ is greater than -1. If G is the geometric mean of $(1 + r_1), (1 + r_2), \cdots, (1 + r_n)$ and $r := G - 1$ then prove that

$$(1 + r_1) \times (1 + r_2) \times \cdots \times (1 + r_n) = (1 + r)^n.$$

Hence find the average compound rate of interest per annum on one time investment of rupees 10000 fetching 8% compound rate of interest for first 3 years, 12% compound rate of interest for next 2 years and 10% compound rate of interest for last 4 years.

(9) Use arithmetic and geometric mean inequality and prove the following statement. $2(1 - x)(1 + x)(1 + x) \leq \frac{64}{27} \ \forall\, x \in [0, 1]$ and reaches this maximum value for $x = \frac{1}{3}$.

(10) If $x > -1$ then prove that $(1 + x)^n \geq 1 + nx \ \forall\, n \in \mathbb{N}$.

(11) Suppose $x, a \in \mathbb{R}$ are positive and $y := \frac{1}{2}\left(x + \frac{a}{x}\right)$. Prove that $y \geq \frac{1}{2}\left(y + \frac{a}{y}\right) \geq \sqrt{a}$.

(12) Prove that $\left(1 + \frac{1}{n}\right)^n \leq \left(1 + \frac{1}{n+1}\right)^{n+1} \ \forall\, n \in \mathbb{N}$.

(13) If L is the perimeter and A is the area of rectangle R then prove that $L^2 \geq 16A$. Further prove that $L^2 = 16A$ if and only if R is square.

(14) Prove that among all rectangles with fixed perimeter L, square has maximum area.

(15) Prove that among all rectangles with fixed area A, square has minimum perimeter.

(16) Use Cauchy-Schwartz inequality and prove given statement.

(i) If $a, b, c \in \mathbb{R}$ are positive such that $a + b + c \leq 3$ then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq 3$.

(ii) Given any $n \in \mathbb{N}$, if each $x_1, x_2, \cdots, x_n \in \mathbb{R}$ is positive then $\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} \frac{1}{x_i}\right) \geq n^2$.

(iii) If $(x_1, y_1), (x_2, y_2)$ are points on unit circle then $|x_1x_2 + y_1y_2| \leq 1$.

(iv) Given any $n \in \mathbb{N}$, if $x_1, x_2, \cdots, x_n \in \mathbb{R}$ and $y_1, y_2, \cdots, y_n \in \mathbb{R}$ then

$$\left(\sum_{i=1}^{n} (x_i + y_i)^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} x_i^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}}.$$

(v) For each $n \in \mathbb{N}$ and each $x \in \mathbb{R}$ we have $|1+x+x^2+\cdots+x^{n-1}| \leq \sqrt{n(1+x^2+x^4+\cdots+x^{2n-2})}$. Further if $x \neq 1$ then

$$\frac{1-x^n}{1-x} \leq \sqrt{\frac{n(1-x^{2n-1})}{1-x^2}}.$$

xxxxxxxxxxxxxx

1.2 Practical 1.2: Hausdorff Property and LUB Axiom of $\mathbb{R}$, Archimedian Property

1.2.1 Prerequisite of Practical 1.2

(1) Let $p, \varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. The $\varepsilon-$ **neighbourhood** of p is the set denoted by $N(p,\varepsilon) = \{x \in \mathbb{R}/|x-p| < \varepsilon\}$.

Let $p, \varepsilon \in \mathbb{R}$ and $\varepsilon > 0$. A **deleted** $\varepsilon-$ **neighbourhood** of p is a set $N(p,\varepsilon) \setminus \{p\}$, where $\varepsilon > 0$.

$$N(p,\varepsilon) = (p-\varepsilon, p+\varepsilon).$$

(2) **Interior point**: Let $D \subseteq \mathbb{R}$. Let $c \in D$. Then c is called an interior point of D if there exists a neighbourhood of c contained in D.
So, $c \in D$ is called an interior point of D if there exists $\varepsilon > 0$ such that $N(c,\varepsilon) \subseteq D$.

(3) **Limit point**: Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is said to be a limit point of D if every neighbourhood of c contains at least one point of D other than c.
That is, $c \in \mathbb{R}$ is a limit point of D if for every $\varepsilon > 0$, there exist $x \in D$ such that $x \in N(c,\varepsilon)$.

(4) **Hausdorff Property**: Any two distinct real numbers have disjoint neighbourhoods.
If $p, q \in \mathbb{R}$ are such that $p \neq q$ then there exist disjoint neighbourhoods of p and q. That is, there exists $r > 0$ such that $N(p,r) \cap N(q,r) = \emptyset$.

(5) Let S be a subset of $\mathbb{R}$. We say that a real number K is an **upper bound** of S if for each $x \in S$ we have $x \leq K$.
Hence, K is an upper bound of S if $\forall\, x \in S$, $x \leq K$.
S is said to be **bounded above** if it has an upper bound in $\mathbb{R}$. That is, S is said to be bounded above if there exists $K \in \mathbb{R}$ such that K is an upper bound of S.
Hence, S is said to be bounded above if there exists $K \in \mathbb{R}$ such that $x \leq K$, for all $x \in S$.
A set $S \subset \mathbb{R}$ is **not bounded above** (or **unbounded above**) if $K \in \mathbb{R}$ then K is not an upper bound of S. That is, for every $K \in \mathbb{R}$, there exists $x \in S$ such that $x > K$.

(6) Let S be a subset of $\mathbb{R}$. We say that a real number k is a **lower bound** of S if for each $x \in S$, we have $k \leq x$.
Hence, k is a lower bound of S if $k \leq x$ for all $x \in S$.
S is said to be **bounded below** if it has a lower bound in $\mathbb{R}$. That is, there exists $k \in \mathbb{R}$ such that k is a lower bound of S.

Hence, S is said to be bounded below if there exists $k \in \mathbb{R}$ such that $k \leq x \in S$ for all $x \in S$.
A set $S \subset \mathbb{R}$ is **not bounded below** (or **unbounded below**) if $k \in \mathbb{R}$ implies k is not a lower bound of S. That is, for every $k \in \mathbb{R}$, there exists $x \in S$ such that $x < k$.

(7) Let S be a subset of $\mathbb{R}$. S is said to be **bounded** if it is bounded above as well as bounded below.

(8) If $S = \emptyset$, then every real number is an upper bound as well as a lower bound of S.

(9) A nonempty set $S \subseteq \mathbb{R}$ is bounded if and only if there is $M \in \mathbb{R}^+$ such that, $|x| \leq M \;\; \forall\, x \in S$.

(10) Let $S \subseteq \mathbb{R}$. $M \in \mathbb{R}$ is called a **least upper bound** or **supremum** of S, denoted by $\sup(S)$, if M satisfies the following two properties:
(i) M is an upper bound of S.
(ii) If K is any upper bound of S then $M \leq K$.

(11) If S has a supremum, then it is unique.

(12) $\emptyset$ does not have a supremum.

(13) Let $S \subseteq \mathbb{R}$. $m \in \mathbb{R}$ is called a **greatest lower bound** or **infimum** of S, denoted by $\inf(S)$, if m satisfies the following two properties:
(i) m is a lower bound of S.
(ii) If k is any lower bound of S then $k \leq m$.

(14) If S has an infimum, then it is unique.

(15) $\emptyset$ does not have an infimum.

(16) **The LUB axiom or Order Completeness of $\mathbb{R}$ or Completeness axiom:**
Every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$.

(17) Every nonempty set S of real numbers that is bounded below has an infimum in $\mathbb{R}$.

(18) Let S be a non-empty subset of $\mathbb{R}$. Then the following statements are true.

 (i) The supremum(or infimum) of S need not be an element of S. That is, $\sup(S)$ (or $\inf S$) may or may not belong to S.
 (ii) If the supremum of a set S is an element of S, then it is called the **maximum** of S, and it is denoted by $\max S$.
 (iii) If the infimum of a set S is an element of S, then it is called the **minimum** of S, and it is denoted by $\min S$.
 (iv) If $K \in \mathbb{R}$ is an upper bound of S and $K \in S$ then $K = \sup S$.
 (v) If $k \in \mathbb{R}$ is a lower bound of S and $k \in S$ then $K = \inf S$.

(19) Characterisation of supremum:
Let S be a nonempty subset of $\mathbb{R}$ that is bounded above. Let $M \in \mathbb{R}$. Then $M = \sup S$ if and only if
(I) M is an upper bound of S.
(II) for any $\varepsilon > 0$, there is an element $a \in S$ such that $M - \varepsilon < a \leq M$.

(20) Characterisation of infimum:
Let S be a nonempty subset of $\mathbb{R}$ that is bounded below. Let $m \in \mathbb{R}$.
Then $m = \inf S$ if and only if
(I) m is a lower bound of S.
(II) for any $\varepsilon > 0$, there is an element $a \in S$ such that $m \le a < m + \varepsilon$.

(21) Let A, B be nonempty subsets of $\mathbb{R}$.

(i) If $A \subseteq B$ and B is bounded then A is also bounded.
(ii) If $A \subseteq B$, and B is bounded, then $\inf B \le \inf A \le \sup A \le \sup B$.
(iii) If A and B are bounded then:

(a) $A \cup B$ and $A \cap B$ are also bounded.
(b) $\inf(A \cup B) = \min\{\inf A,\ \inf B\}$.
(c) $\sup(A \cup B) = \max\{\sup A,\ \sup B\}$.
(d) $\inf(A \cap B) \ge \max\{\inf A,\ \inf B\}$.
(e) $\sup(A \cap B) \le \min\{\sup A,\ \sup B\}$.

(22) For $a \in \mathbb{R}$, define the set $aS = \{ax \ : x \in S\}$.
If S is bounded then aS is a bounded set and

(i) $\sup aS = \begin{cases} a \sup S, & \text{if } a \ge 0 \\ a \inf S, & \text{if } a < 0 \end{cases}$ (ii) $\inf aS = \begin{cases} a \inf S, & \text{if } a \ge 0 \\ a \sup S, & \text{if } a < 0 \end{cases}$

(23) For $a \in \mathbb{R}$, define $a + S = \{a + x : x \in S\}$.
If S is bounded then the set $a + S$ is bounded and

(i) $\sup(a + S) = a + \sup S$. (ii) $\inf(a + S) = a + \inf S$

(24) Define the set $C = \{a + b : a \in A, b \in B\}$.
If A and B are bounded then

(i) C is a bounded set. (ii) $\sup C = \sup A + \sup B$. (iii) $\inf C = \inf A + \inf B$.

(25) **Archimedean property**
Given any $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $x < n$.

(26) Let $x, y \in \mathbb{R}$.

(i) If $x > 0$ then there exists $n \in \mathbb{N}$ such that $0 < \dfrac{1}{n} < x$.
(ii) If $x > 0$ then there exists $n \in \mathbb{N}$ such that $y < nx$.
(iii) If $x > 0$ then there exists $m \in \mathbb{N}$ such that $m - 1 \le x < m$.

(27) **Density of rationals**: If x and y are any real numbers with $x < y$, then there exists a rational number r such that $x < r < y$. Moreover this number r can always be selected so that it is nonzero.

(28) If x and y are any real numbers with $x < y$, then there exists an irrational number s such that $x < s < y$.

1.2

1.2.2 PRACTICAL 1.2

(A) Objective Questions

Choose correct alternative in each of the following:

(1) Every non-empty subset of $\mathbb{R}$ which is bounded above has ....

(a) supremum in $\mathbb{R}$
(b) supremum and infimum in $\mathbb{R}$
(c) greatest lower bound in $\mathbb{R}$
(d) neither supremum nor infimum in $\mathbb{R}$

(2) If A is interval $[-2, 1)$ then ....

(a) $\inf A \in A$ and $\sup A \in A$
(b) $\inf A \in A$ and $\sup A \notin A$
(c) $\inf A \notin A$ and $\sup A \notin A$
(d) $\inf A \notin A$ and $\sup A \in A$

(3) $A \subseteq \mathbb{R}$ and $M \in \mathbb{R}$ is such that $M = \sup A$. If $B := \{-5x \mid x \in A\}$ then the infimum of set B is ....

(a) $5M$ (b) $-5M$ (c) $\frac{M}{5}$ (d) None of these

(4) If X is a finite subset of $\mathbb{R}$ then ....

(a) X is bounded above but not bounded below
(b) X is bounded below but not bounded above
(c) X is bounded
(d) None of these

(5) If .... then intervals $(1.5 - \epsilon, 1.5 + \epsilon)$ and $(3.4 - \epsilon, 3.4 + \epsilon)$ are disjoint.

(a) $0 < \epsilon \leq 3.4$ (b) $0 < \epsilon \leq 1$ (c) $0 < \epsilon \leq 0.95$ (d) $0 < \epsilon \leq 1.5$

(6) If l_1, l_2 are two distinct lower bounds of set $S \subseteq \mathbb{R}$ and if $l_1 \in S$ then ....

(a) $l_2 \notin S$ (b) $l_2 \in S$ (c) $l_1 < l_2$ (d) None of these

(7) If u_1, u_2 are two distinct upper bounds of set $S \subseteq \mathbb{R}$ and if $u_1 \in S$ then ....

(a) $u_2 \notin S$ (b) $u_2 \in S$ (c) $u_2 < u_1$ (d) None of these

(8) If the set U of all upper bounds of set $S \subseteq \mathbb{R}$ is non-empty then U is ....

(a) finite set
(b) bounded below
(c) bounded above
(d) None of these

(9) If the set L of all lower bounds of set $S \subseteq \mathbb{R}$ is non-empty then L is ....

(a) finite set
(b) bounded below
(c) bounded above
(d) None of these

(10) If $A, B \subseteq \mathbb{R}$ with $\inf A = -2$ and $\inf B = -3$ then $\inf (A \cup B) =$ ....

(a) -3 (b) -2 (c) -5 (d) None of these

(11) If $A, B \subseteq \mathbb{R}$ with $\inf A = -2$ and $\inf B = -3$ then $\inf \{a + b \mid a \in A \text{ and } b \in B\} =$ ....

(a) -3 (b) -2 (c) -5 (d) None of these

(12) $A = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$ then $\sup A$ is

(a) 1 (b) 0 (c) does not exist (d) $\frac{3}{2}$

(13) $A = \left\{1 + \frac{(-1)^n}{n} : n \in \mathbb{N}\right\}$ then $\inf A$ is

(a) 1 (b) -2 (c) does not exist (d) $\frac{-3}{2}$

(14) $A = \left\{-1 + \frac{1}{n} : n \in \mathbb{N}\right\}$ then $\inf A$ is

(a) 1 (b) -1 (c) does not exist (d) $\frac{-3}{2}$

(15) $A = \left\{1 + \frac{1}{n} : n \in \mathbb{N}\right\}$ then $\sup A$ is

(a) 2 (b) 0 (c) does not exist (d) $\frac{3}{2}$

(16) $A = \{x \in \mathbb{R} : |x - 3| < |x + 5|\}$ then

(a) $\sup A$ does not exist and $\inf A = -1$
(b) $\sup A = 1$ and $\inf A = 1$
(c) $\sup A = 1$ and $\inf A$ does not exist
(d) None of these

(17) $A = \left\{-1, \frac{-1}{2}, \frac{-1}{3}, \dots\right\}$ then

(a) $\sup A$ does not exist and $\inf A = -1$
(b) $\sup A = -1$ and $\inf A$ does not exist
(c) $\sup A = -1$ and $\inf A$ does not exist
(d) None of the above

(18) $A = \{3, 9, 12, 15, 18\}$ then

(a) $\sup A$ does not exist and $\inf A = 3$
(b) $\sup A = 18$ and $\inf A = 3$
(c) $\sup A = 0$ and $\inf A = -1$
(d) None of the above.

1.2

(19) If $S \subseteq \mathbb{R}$, $M = \sup S$ and $L < M$, then there exists $x \in S$ such that ....

(a) $L < x < M$ (b) $L \leq x < M$ (c) $L < x \leq M$ (d) None of these

(20) If $A = \{1,\ 5,\ 7,\ 11, 14\}$ then

(a) $\inf A \notin A$ and $\sup A \notin A$
(b) $\inf A \notin A$ and $\sup A \in A$
(c) $\inf A \in A$ and $\sup A \notin A$
(d) $\inf A \in A$ and $\sup A \in A$

(21) Between any two distinct real numbers there always exists

(a) only rational numbers
(b) only irrational numbers
(c) both rational and irrational numbers
(d) None of the above

(22) Every non-empty subset of $\mathbb{R}$ which is bounded below has

(a) infimum in $\mathbb{R}$
(b) supremum in $\mathbb{R}$
(c) infimum and supremum in $\mathbb{R}$
(d) neither infimum nor supremum in $\mathbb{R}$

(23) If $m = \inf B$ and $m \in B$ then ....

(1) m is the minimum of Y
(2) m is the maximum of the set of all lower bounds of Y
(3) m is the least upper bound of Y
(4) None of these

(24) If $M = \sup B$ and $M \in B$ then ....

(1) m is the maximum of Y
(2) m is the minimum of the set of all lower bounds of Y
(3) m is the greatest lower bound of Y
(4) None of these

(25) If $\inf B = \sup B$ then ....

(1) Y is empty set
(2) Y is singleton set
(3) Y has only two elements
(4) None of these

(26) For interval $A =$.......... we have $\inf A \in A$ and $\sup A \notin A$.

(1) $(0, 1)$ (2) $(0, 1]$ (3) $[0, 1]$ (4) $[0, 1)$

(B) Descriptive Questions

(1) Find disjoint neighbourhoods of x and y.

(i) $x = 7, y = 8$ (iii) $x = \sqrt{2}, y = \sqrt{3}$ (v) $x = 9, y = 9.5$
(ii) $x = e, y = \pi$ (iv) $x = \frac{2}{3}, y = \frac{3}{2}$.

(2) Decide whether the following sets are bounded above/below (hence bounded). Find the infimum, supremum, maximum and minimum of the given sets if they exists.

(i) $\{x \in \mathbb{R} \mid x^2 - 3x + 2 = 0\}$
(ii) $\{x \in \mathbb{R} \mid x^2 > 2\}$
(iii) $\{3 + \frac{2}{n} \mid n \in \mathbb{N}\}$
(iv) $\{x \in \mathbb{R} \mid x^2 - x - 12 < 0\}$
(v) $\{n + \frac{1}{n} \mid n \in \mathbb{N}\}$
(vi) $\{\mathrm{x} \in \mathbb{R} \mid \mathrm{x}^2 - 5\mathrm{x} + 4 < 0\}$
(vii) $\left\{\frac{x}{x+1} \mid x > 0\right\}$
(viii) $\left\{(-1)^n \left(1 + \frac{1}{n}\right) \mid n \in \mathbb{N}\right\}$
(ix) $\{2^{1|\mathrm{n}} \mid \mathrm{n} \in \mathbb{N}\}$
(x) $\{\frac{1}{\mathrm{x}} \mid 1 = \mathrm{x} < 2\}$
(xi) $\left\{\frac{1+(-1)^n}{n} \,\middle|\, n \in \mathbb{N}\right\}$
(xii) $\{\sin\frac{\mathrm{n}\pi}{2} \mid \mathrm{n} \in \mathbb{N}\}$

(3) Find the maximum, minimum, infimum and supremum of the following sets (whenever exist). Also state whether the given sets contain their infimum and supremum.

(i) $A = \{x \in \mathbb{R} : x^2 - 2x - 15 < 0\}$
(ii) $C = \{1 + (-1)^n : n \in \mathbb{N}\}$
(iii) $B = \{x \in \mathbb{R} : x^2 - 7x + 12 > 0\}$
(iv) $D = \{\cos x : 0 < x < \pi\}$

(4) If $x, y \in \mathbb{R}$ are such that $|x - y| < \frac{1}{n}$ $\forall\, n \in \mathbb{N}$, then show that $x = y$.

(5) If a and b are two real number such that $a < b + \varepsilon$, $\forall\, \varepsilon > 0$, then prove that $a \leq b$.

(6) Show that, if $a \in \mathbb{R}$ such that $|a| < \frac{1}{n}$, $\forall\, n \in \mathbb{N}$, prove that $a = 0$.

(7) Show that if $S = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ then $\inf(S) = 0$

(8) Show that for $a \in \mathbb{R}$, if $S = \{a + \frac{1}{n} \mid n \in \mathbb{N}\}$, then $\inf(S) = a$.

(9) If m is a lower bound of $S \subseteq \mathbb{R}$ such that m $\in$S then prove that m = inf S. If M is an upper bound of $S \subseteq \mathbb{R}$ such that M $\in$S then prove that M = sup S.

(10) Prove that if the set of upper bounds of $S \subseteq \mathbb{R}$ is non-empty, then it is bounded below. Prove that if the set of lower bounds of $S \subseteq \mathbb{R}$ is non-empty, then it is bounded above.

(11) Give an example of set $A \subseteq \mathbb{R}$ such that:

(i) A is bounded above but not below.
(ii) A contains its lub but not its glb.
(iii) A does not contain its lub as well as its glb.
(iv) lub A = glb A.
(v) each element of A is rational but its lub is irrational.
(vi) each element of A is irrational but its lub is rational.

1.2

(12) Exhibit three upper bounds for $\{x \in \mathbb{R} : x \leq 0\}$ and three lower bounds for $\{x \in \mathbb{R} : x \geq 0\}$.

(C) Some More Descriptive Questions

(1) Show that no upper bound or lower bound of the set $\{x \in \mathbb{R} : 0 < x < 1\}$ that belongs to the set.

(2) Let $a \in \mathbb{R}$ be positive and $S := \{t \in \mathbb{R} \mid t > 0 \text{ and } t^2 < a\}$.
Then prove that:

(i) $\frac{a}{1+a}$ is an element of S.

(ii) $1 + a$ is an upper bound for S.

(iii) $\sup S$ exists in $\mathbb{R}$.

(iv) Put $u := \sup S$. Suppose that $u^2 < a$ then there exists $h > 0$ such that $(u+h)^2 - u^2 < a - u^2$. Which is a contradiction to u is an upper bound of S.

(v) Suppose that $u^2 > a$ then there exists $h > 0$ such that $u^2 - (u - h)^2 < u^2 - a$. Which is a contradiction to u is an least upper bound of S.

(vi) $u^2 = a$.

(3) Let $a \in \mathbb{R}$ be positive and $S := \{t \in \mathbb{R} \mid t^n < a\}$. Then prove that:

(i) $ca1 + a$ is an element of S and $\sup S \in \mathbb{R}$ and

(ii) $(\sup S)^n = a$.

(iii) If $S = \{a_n \in \mathbb{R} \mid n \in \mathbb{N}\}$ is bounded and $A_n := \{a_k \in S \mid k \geq n\}\ \forall\, n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, prove that $\inf A_n \leq \inf A_{n+1} \leq \sup A_{n+1} \leq \sup A_n$.

xxxxxxxxxxxxxxx

1.3 Practical 1.3: Convergence and divergence of sequences, bounded sequences, Sandwich Theorem.

1.3.1 Prerequisite of Practical 1.3

(1) A **sequence** of real numbers is a function whose domain is the set $\mathbb{N}$ and codomain is the set $\mathbb{R}$.
A sequence is denoted by $(x_n), (y_n), (a_n), (b_n)$, and so on.
The value of a sequence (x_n) at $n \in \mathbb{N}$ is given by x_n, and it is called the n^{th} **term** of that sequence.
The set $\{x_n : n \in \mathbb{N}\}$ is called the **set of terms** of the sequence (x_n).
Note: Although a sequence has infinitely many terms, the set of its terms need not be infinite. Consider the following sequences (x_n) whose n^{th} term is defined by

(i) $x_n = 1$. So, $\{a_n : n \in \mathbb{N}\} = \{1\}$. The set of the terms of the sequence is finite.

(ii) $x_n = (-1)^n$. So, $\{a_n : n \in \mathbb{N}\} = \{-1, 1\}$. The set of the terms of the sequence is finite.

(iii) $x_n = n^2$. So, $\{a_n : n \in \mathbb{N}\} = \{1, 4, 9, \cdots\}$. The set of the terms of the sequence is infinite.

(2) A sequence (x_n) is said to be a **convergent** sequence if there is $p \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - p| < \varepsilon$ for all $n \geq n_0$.

In this case, we say that (x_n) **converges** to p or that p is a **limit** of (x_n), and write as $x_n \longrightarrow p$ (as $n \longrightarrow \infty$) or $\lim\limits_{n \longrightarrow \infty} x_n = p$.

(3) We write $x_n \not\longrightarrow p$ if the sequence (x_n) does not converge to p.

(4) A sequence that is not convergent is said to be **divergent**.

(5) Some of the convergent sequences are

(i) $x_n = \dfrac{1}{n}$ (ii) $x_n = \dfrac{(-1)^n}{n}$ (iii) $x_n = \sqrt{n+3} - \sqrt{n}$

We show that (ii) is a convergent sequence. We can observe that $\dfrac{(-1)^n}{n} \to 0$. We need to prove the following: $\forall\, \varepsilon > 0,\ \exists\, n_0 \in \mathbb{N}$ such that $\forall\, n \geq n_0$, $\left|\dfrac{(-1)^n}{n} - 0\right| < \varepsilon$.
Let $\varepsilon > 0$ be given.
For all $n \in \mathbb{N}$, we have,

$$|x_n - p| = \left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n}.$$

Since $\varepsilon > 0$, by Archimedean Property, we get, $n_0 \in \mathbb{N}$ such that $\dfrac{1}{\varepsilon} < n_0$.

$$\frac{1}{\varepsilon} < n_0 \Longrightarrow \frac{1}{n_0} < \varepsilon \qquad \cdots\cdots \text{(I)}$$

Hence, for all $n \geq n_0$, we have, $\dfrac{1}{n} \leq \dfrac{1}{n_0}$.

$$\Longrightarrow |x_n - p| = \left|\frac{(-1)^n}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon. \qquad \cdots \text{(from (I))}$$

So, $\dfrac{(-1)^n}{n} \longrightarrow 0$.

(6) Some of the divergent sequences are as follows.

(i) $x_n = n$ (ii) $x_n = (-1)^n$ (iii) $x_n = (-1)^n n$

(7) A convergent sequence converges to a unique limit.

(8) We say that (x_n) **tends** to ∞ or diverges to ∞ and write as $\lim\limits_{n \longrightarrow \infty} x_n = \infty$ if for every $M \in \mathbb{R}$, there exists $n_0 \in \mathbb{N}$ such that $x_n > M$ for all $n \geq n_0$.

1.3

(9) We say that (x_n) **tends** to $-\infty$ or diverges to $-\infty$ and write as $\lim_{n\longrightarrow\infty} x_n = -\infty$ if for every $M \in \mathbb{R}$, there exists $n_0 \in \mathbb{N}$ such that $x_n < M$ for all $n \geq n_0$.

(10) If the sequence (x_n) diverges but does not diverge to infinity and does not diverge to minus infinity, then (x_n) is said to **oscillate.**
Some of the oscillating sequences are as follows.

(i) $(-1)^n$ (ii) $\sin\left(\frac{n\pi}{2}\right)$ (iii) $(-1)^n\left(1+\frac{1}{n}\right)^n$ (iv) $1, 2, 1, 3, 1, 4, \ldots$

(11) A sequence (x_n) is said to be **bounded above** if $\exists\ K \in \mathbb{R}$ such that $x_n \leq K,\ \forall\ n \in \mathbb{N}$.
A sequence (x_n) is said to be **bounded below** if $\exists\ k \in \mathbb{R}$ such that $k \leq x_n,\ \forall\ n \in \mathbb{N}$.
A sequence (x_n) is said to be **bounded** if (x_n) is bounded above and bounded below.
A sequence is said to be **unbounded** if it is not bounded. That is, it is not bounded above or not bounded below or not bounded above as well as not bounded below.
Some examples of bounded and unbounded sequences:

(i) $x_n = c,\ \forall\ n \in \mathbb{N}$ where $c \in \mathbb{R}$. (bounded sequence)
(ii) $x_n = \frac{1}{n},\ \forall\ n \in \mathbb{N}$ (bounded sequence)
(iii) $y_n = n,\ \forall\ n \in \mathbb{N}$(bounded below but not bounded above, so unbounded sequence)
(iv) $z_n = (-1)^n,\ \forall\ n \in \mathbb{N}$ (bounded sequence)

(12) Every convergent sequence is bounded. (Converse not true as $x_n = (-1)^n$ is bounded but not congergent)

(13) Algebra of Convergent Sequences: Let $x_n \longrightarrow p$ and $y_n \longrightarrow q$. Then

(i) $x_n + y_n \longrightarrow p + q$,
(ii) $x_n - y_n \longrightarrow p - q$,
(iii) $rx_n \longrightarrow rp$ for any $r \in \mathbb{R}$,
(iv) $x_n y_n \longrightarrow pq$,
(v) If $x_n \neq 0\ \forall\ n \in \mathbb{N}$ and $p \neq 0$ then $\dfrac{1}{x_n} \longrightarrow \dfrac{1}{p}$.
(vi) If $x_n \longrightarrow p$ and $y_n \longrightarrow q, y_n \neq 0\ \forall\ n \in \mathbb{N}$ and $q \neq 0$ then $\dfrac{x_n}{y_n} \longrightarrow \dfrac{p}{q}$.

(14) Properties of Convergent Sequences:

(i) If $x_n \longrightarrow p$ then $|x_n| \longrightarrow |p|$. (Converse not true.)
(ii) $x_n \longrightarrow 0$ if and only if $|x_n| \longrightarrow 0$.
(iii) (x_n) is bounded and $y_n \longrightarrow 0$ then $x_n\ y_n \longrightarrow 0$.
(iv) If $x_n \geq 0\ \ \forall\ n \in \mathbb{N}$ and $x_n \longrightarrow p$ then $p \geq 0$ and $x_n^{\frac{1}{k}} \longrightarrow p^{\frac{1}{k}}$ for any $k \in \mathbb{N}$.
(v) If $x_n \longrightarrow p$ where $p > 0$ then $\exists\ n_0 \in \mathbb{N}$ such that $x_n > 0, \forall\ n \geq n_0$.
(vi) If $x_n \longrightarrow p$ and $p \neq 0$ then $\exists\ m \in \mathbb{N}$ such that $x_n \neq 0, \forall\ n \geq m$.
(vii) Suppose $x_n \longrightarrow p$ and $y_n \longrightarrow q$. If there is $n_0 \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq n_0$, then $p \leq q$.
(8) Suppose $x_n \longrightarrow p$ and $y_n \longrightarrow q$. If $p < q$, then there is $m_0 \in \mathbb{N}$ such that $x_n < y_n$ for all $n \geq m_0$.

(15) **Sandwich Theorem for sequences**: Let $(x_n), (y_n)$ and (z_n) be sequences and $p \in \mathbb{R}$ be such that $x_n \le z_n \le y_n \ \forall\ n \in \mathbb{N}$ and $x_n \longrightarrow p$ as well as $y_n \longrightarrow p$. Then $z_n \longrightarrow p$.

(16) Convergence of standard sequences:

(1) $\lim\limits_{n\longrightarrow\infty} \dfrac{1}{1+na} = 0 \quad \forall\ a > 0.$

(2) $\lim\limits_{n\longrightarrow\infty} b^n = 0 \quad \forall\ b,\ |b| < 1.$

(3) $\lim\limits_{n\longrightarrow\infty} c^{\frac{1}{n}} = 1, \quad \forall\ c > 0.$

(4) $\lim\limits_{n\longrightarrow\infty} n^{\frac{1}{n}} = 1.$

(17) Let $a \in \mathbb{R}$. We will show that there exists a sequence of rationals converging to a.
For $n = 1, a < a+1$. By Density theorem, there exists a rational say x_1 between a and $a+1$.
For $n = 2, a < a + \frac{1}{2}$. By Density theorem, there exists a rational say x_2 between a and $a + \frac{1}{2}$.
We continue this way, and for $n = k$, by Density theorem, there exists a rational say x_k between a and $a + \frac{1}{k}$.
For $n = k+1$, by Density theorem, there exists a rational say x_{k+1} between a and $a + \frac{1}{k+1}$.
Hence, there exists a rational x_n between a and $a + \frac{1}{n}$ for all $n \in \mathbb{N}$.
Thus, $a < x_n < a + \frac{1}{n}$ for all $n \in \mathbb{N}$. This implies $0 < x_n - a < \frac{1}{n}$ for all $n \in \mathbb{N}$.
By Sandwich theorem, $x_n - a \longrightarrow 0$ and hence $x_n \longrightarrow a$.
Thus, we know that there exists a sequence of rationals converging to a where a is any real number.
We may want to find one such actual sequence.
Let $a \in \mathbb{R}$.

Case 1 $a > 0$. Let $n \in \mathbb{N}$. Consider na. Clearly $na > 0$.
By prerequisite of practical 1.1, no. (26), there exists $m \in \mathbb{N}$ such that $m - 1 \le na < m$.
So, $m - 1 \le na < m < m + 1$. This implies $-1 \le na - m < 1$. ($m = [na]$ integral part of na)
That is, $-\frac{1}{n} \le a - \frac{m}{n} < \frac{1}{n}$ for all $n \in \mathbb{N}$.
Hence $\left|\frac{m}{n} - a\right| < \frac{1}{n}$.
As, $\frac{1}{n} \longrightarrow 0$ and $0 \le \left|\frac{m}{n} - a\right|$, we have, $\left|\frac{m}{n} - a\right| \longrightarrow 0$ and hence $\frac{m}{n} \longrightarrow a$.
Since $\frac{m}{n}$ where $m = [na]$ is a rational for every $n \in \mathbb{N}$, we have found a sequence of rationals converging to a when $a > 0$.

Case 2 $a < 0$. Then $-a > 0$. By above case, we have a sequence say (x_n) of rationals converging to $-a$. Hence $(-x_n)$ is a sequence of rationals converging to a.

Case 3 $a = 0$. Then consider the sequence $x_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. This is a sequence of rationals converging to 0. Even the constant sequence $x_n = 0$ for all $n \in \mathbb{N}$ is also another example of a sequence of rationals converging to a.

1.3

(18) Let $a \in \mathbb{R}$. We will show that there exists a sequence of irrrationals converging to a.
For $n = 1, a < a + 1$. From Density theorem, there exists an irrational say y_1 between a and $a + 1$.
For $n = 2, a < a + \frac{1}{2}$. From Density theorem, there exists an irrational say y_2 between a and $a + \frac{1}{2}$.
We continue this way, and for $n = k$, from Density theorem, there exists an irrational say y_k between a and $a + \frac{1}{k}$.
For $n = k + 1$, from Density theorem, there exists an irrational say y_{k+1} between a and $a + \frac{1}{k+1}$.
Hence, there exists an irrational x_n between a and $a + \frac{1}{n}$ for all $n \in \mathbb{N}$.
Thus, $a < x_n < a + \frac{1}{n}$ for all $n \in \mathbb{N}$. This implies $0 < x_n - a < \frac{1}{n}$ for all $n \in \mathbb{N}$.
By Sandwich theorem, $x_n - a \longrightarrow 0$ and hence $x_n \longrightarrow a$.
Thus, we know that there exists a sequence of irrationals converging to a where a is any real number.
We may want to find one such actual sequence.
Let $a \in \mathbb{R}$. We have seen that we can find a sequence of rationals say (x_n) converging to a.
Now, $\left(\frac{\sqrt{2}}{n}\right)$ is a sequence of irrationals converging to 0.
So, $\left(x_n + \frac{\sqrt{2}}{n}\right)$ is a sequence of irrationals converging to a.

1.3.2 PRACTICAL 1.3

(A) Objective Questions

Choose correct alternative in each of the following:

(1) Given $\lim_{n \longrightarrow \infty} x_n = 5$, $\lim_{n \longrightarrow \infty} y_n = -3$, then $\lim_{n \longrightarrow \infty} \frac{1 + \sqrt{x_n}}{1 + y_n^2}$ is

(a) $\frac{1 + \sqrt{5}}{7}$ (b) $\frac{6}{7}$ (c) $\frac{1 + \sqrt{5}}{10}$ (d) does not exist.

(2) If $x_n = \frac{4n^2 - 3n + 2}{n^2 + 5n}$ for all $n \in \mathbb{N}$, then

(a) (x_n) is not bounded below.
(b) (x_n) is not bounded above.
(c) (x_n) is not convergent.
(d) none of these.

(3) Which of the following sequence is divergent?

(a) $(n^{\frac{1}{n}})$ (b) $(2^{\frac{1}{n}})$ (c) $(\sqrt[n]{n!})$ (d) none of these

(4) If $(x_n + y_n)$ and (y_n) are convergent sequences of real numbers then (x_n)

(a) is divergent.

(b) may or may not be convergent.

(c) is convergent.

(d) none of the above.

(5) (x_n) and (y_n) are sequences of real numbers and $a_n = |x_n - y_n|$ for all $n \in \mathbb{N}$. If (x_n) converges to p and (a_n) converges to 0 then

(a) (y_n) is convergent but $\lim\limits_{n\longrightarrow\infty} y_n \neq p$.

(b) (y_n) may not be convergent.

(c) (y_n) is convergent and $\lim\limits_{n\longrightarrow\infty} y_n = p$.

(d) none of the above.

(6) (x_n) and (y_n) are sequences of real numbers such that $x_n < y_n$ for all $n \in \mathbb{N}$.

(a) $\lim\limits_{n\longrightarrow\infty} x_n < \lim\limits_{n\longrightarrow\infty} y_n$.

(b) $\lim\limits_{n\longrightarrow\infty} x_n \geq \lim\limits_{n\longrightarrow\infty} y_n$.

(c) $\lim\limits_{n\longrightarrow\infty} x_n = \lim\limits_{n\longrightarrow\infty} y_n$.

(d) $\lim\limits_{n\longrightarrow\infty} x_n \leq \lim\limits_{n\longrightarrow\infty} y_n$.

(7) (x_n) and (y_n) are sequences such that $(x_n + y_n)$ is convergent. Then

(a) at least one of (x_n) and (y_n) is convergent.

(b) both (x_n) and (y_n) are convergent.

(c) both (x_n) and (y_n) may be divergent.

(d) none of these.

(8) Which one of the following statement is not true?

(a) A convergent sequence converges to a unique limit.

(b) Every convergent sequence is bounded.

(c) A bounded sequence is convergent.

(d) none of these.

(9) (x_n) and (y_n) are sequences. Under which of the following condition is the sequence $(x_n * y_n)$ convergent?

(a) (x_n) is convergent.

(b) (x_n) is convergent and (y_n) is bounded.

(c) (x_n) converges to 0 and (y_n) bounded.

(d) none of these.

(10) $x_1 = 1$ and $x_{n+1} = \sqrt{14 + 5x_n}$. Assume that the sequence (x_n)

(a) $\sqrt{19}$

(b) -2

(c) 7

(d) 0

(B) Descriptive Questions

(1) The sequence (x_n) is defined by the following formula for the n^{th} term. Write first five terms in each case.

(i) $x_n = \dfrac{1}{n^n}$ (ii) $\dfrac{(-1)^n}{n}$ (iii) $x_n = \cos\left(\frac{n\pi}{2}\right)$ (iv) $x_n = \dfrac{n+1}{n\sqrt{n}}$ (v) $x_n = \left(1 + \frac{1}{n}\right)^n$.

(2) List the first four terms of the following inductively defined sequences.

(i) $x_1 = 2, x_{n+1} = 5x_n - 3$.

(ii) $x_1 = 1, x_{n+1} = x_n + \dfrac{1}{x_n}$.

(iii) $x_1 = 7, x_2 = 5, x_{n+2} = x_{n+1} + x_n$.

(3) Find a value of n_0 for the following convergent sequences (x_n) and for the given ϵ so that $|x_n - p| < \epsilon$ for every $n \geq n_0$ where p is the limit of the sequence (x_n).

(i) $x_n = \dfrac{1}{n}$ for all $n \in \mathbb{N}$ and $\epsilon = 0.004$.

(ii) $x_n = \dfrac{1}{n}$ for all $n \in \mathbb{N}$ and $\epsilon = 0.05$.

(iii) Compare the values of n_0 that you have found in the above two examples. Write your conclusion that will give the relation between the value of ϵ and corresponding value of n_0 for the given convergent sequence.

(iv) $x_n = \dfrac{1}{n^2}$ for all $n \in \mathbb{N}$ and $\epsilon = 0.0132$.

(v) $x_n = \dfrac{1}{n^2}$ for all $n \in \mathbb{N}$ and $\epsilon = 0.05$.

(vi) $x_n = \dfrac{2n+1}{3n+2}$ for all $n \in \mathbb{N}$ and $\epsilon = 0.0132$.

(vii) $x_n = \dfrac{2n+1}{3n+2}$ for all $n \in \mathbb{N}$ and $\epsilon = 0.05$.

(4) Use the $\epsilon - n_0$ definition of the limit of a sequence to establish the following limits.

(i) $\lim\limits_{n\longrightarrow\infty} \dfrac{3n}{n+2} = 3$.

(ii) $\lim\limits_{n\longrightarrow\infty} \dfrac{n^2-2}{2n^2+3} = \dfrac{1}{2}$.

(iii) $\lim\limits_{n\longrightarrow\infty} 1 + \dfrac{(-1)^n}{n} = 1$.

(iv) $\lim\limits_{n\longrightarrow\infty} \dfrac{1+2\sqrt{n}}{1+\sqrt{n}} = 2$.

(v) $\lim\limits_{n\longrightarrow\infty} \dfrac{3n^2-4n+2}{4n^2+2n-1} = \dfrac{3}{4}$.

(vi) $\lim\limits_{n\longrightarrow\infty} \sqrt{n+2} - \sqrt{n} = 0$.

(vii) $\lim\limits_{n\longrightarrow\infty} \dfrac{(0.5)^n}{n!} = 0$

(5) Use Sandwich Theorem (Squeeze theorem) to show that each of the following sequence is convergent and also find its limit using the Sandwich Theorem.

(i) $x_n = \dfrac{(-1)^n \sin n}{2n}$

(ii) $x_n = \dfrac{7^n}{n!}$.

(iii) $x_n = \dfrac{n^2 + n\sin n}{n^2 + \cos n}$.

(iv) $x_n = \left(1 + \dfrac{n}{n+1}\right)^{\frac{1}{n}}$.

(v) $x_n = \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+2)^2} + \cdots . + \dfrac{1}{(n+n)^2}$.

(vi) $x_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}}$ (vii) $x_n = \left(1+\frac{1}{n^2}\right)^n$

(6) If $\lim\limits_{n\longrightarrow\infty} x_n = 3$ and $\lim\limits_{n\longrightarrow\infty} y_n = 4$ then prove that there exists $n_0 \in \mathbb{N}$ such that $x_n+y_n < 8$, for all $n \geq n_0$.

(7) Give an example of each of the following.

(i) A bounded sequence which is not convergent.

(ii) A sequence (x_n) converges to 0 but $(x_n y_n)$ does not converge to 0.

(8) $x_n = \begin{cases} 3 - \frac{1}{n} & \text{if } n \text{ is odd,} \\ 4 + \frac{1}{n^2} & \text{if } n \text{ is even.} \end{cases}$
Is (x_n) convergent? Justify your answer.

(9) Give an example of a divergent sequence (x_n) for which $x_n \nrightarrow \infty$ and $x_n \nrightarrow -\infty$ and which is

(i) bounded (ii) unbounded

1.4 Practical 1.4: Monotonic sequences, Cauchy sequences, Subsequences.

1.4.1 Prerequisite of Practical 1.4

(1) A sequence (x_n) is said to be **monotonically increasing** if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$.
A sequence (x_n) is said to be **monotonically decreasing** if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$.
A sequence is said to be **monotonic** if it is either monotonically increasing or monotonically decreasing.
Note:

(i) If a sequence (x_n) is monotonically increasing then it is bounded below by x_1.

(ii) If a sequence (x_n) is monotonically decreasing then it is bounded above by x_1.

(iii) A constant sequence is both, monotonically increasing as well as monotonically decreasing.

(iv) To check whether the sequence (x_n) is monotonically increasing or decreasing, we need to consider the difference $x_{n+1} - x_n$. If $x_{n+1} - x_n \geq 0$ then (x_n) is monotonically increasing and if $x_{n+1} - x_n \leq 0$ then (x_n) is monotonically decreasing.

(2) If a monotonically increasing sequence (x_n) is bounded above then it is convergent and it converges to the supremum of $\{x_n : n \in \mathbb{N}\}$.

(3) If a monotonically decreasing sequence (x_n) is bounded below then it is convergent and it converges to the infimum of $\{x_n : n \in \mathbb{N}\}$.

(4) A monotonic sequence is convergent if and only if it is bounded.

(5) $\lim_{n \longrightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$

(6) Let (x_n) be a sequence. If $n_1, n_2, \cdots$ are positive integers such that $n_k < n_{k+1}$ for each $k \in \mathbb{N}$, then the sequence (x_{n_k}) whose terms are $x_{n_1}, x_{n_2}, \cdots$ is called a **subsequence** of (x_n).
Note:

(i) Every sequence is it's own subsequence.

(ii) Since $n_{k+1} > n_k$ for all $k \in \mathbb{N}$, we have $n_k \geq k$ for all $k \in \mathbb{N}$.

(iii) $n_k \longrightarrow \infty$ as $k \longrightarrow \infty$.

(7) Let (x_n) be a sequence and (x_{n_k}) be a subsequence of (x_n). Let $p \in \mathbb{R}$. We say that the **subsequence** (x_{n_k}) **converges** to p and write as $x_{n_k} \longrightarrow p$, if for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that $|x_{n_k} - p| < \varepsilon$, for all $k \geq k_0$.

(8) A subsequence of a convergent sequence is convergent and it converges to the same limit as that of the sequence.
Note:

(i) A sequence (x_n) converges to p if and only if every subsequence of (x_n) converges to p.

(ii) If a sequence has two subsequences converging to two distinct limits then the given sequence is not convergent.

(iii) A sequence (x_n) tends to ∞ if and only if every subsequence of (x_n) tends to ∞.

(iv) A sequence (x_n) tends to $-\infty$ if and only if every subsequence of (x_n) tends to $-\infty$.

(9) A sequence (x_n) is called a **Cauchy sequence** if for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq n_0$.

(10) Every Cauchy sequence is bounded.(converse not true)

(11) Every convergent sequence is Cauchy.

(12) Every sequence in $\mathbb{R}$ has a monotonic subsequence. (only statement)

(13) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. (only statement)

(14) every Cauchy sequence in $\mathbb{R}$ is convergent.

1.4.2 PRACTICAL 1.4

(A) Objective Questions

Choose correct alternative in each of the following:

(1) $\lim_{n \longrightarrow \infty} \left(1 - \frac{7}{n}\right)^n$ is

(a) e^7 (b) $\frac{7}{e}$ (c) $\frac{7}{e^7}$ (d) does not exist.

(2) $x_1 = 2$ and $x_2 = 4$ and $x_n = \frac{x_{n-1}}{2n} + (2n-1)\frac{x_{n-2}}{n}$ for all $n \geq 3$. If $\lim_{n \longrightarrow \infty} x_n$ exists then $\lim_{n \longrightarrow \infty} x_n$ is

(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) does not exist.

(3) If x_1 and x_2 are positive and $x_{n+2} = \sqrt{x_n} + \sqrt{x_{n+1}}$ for all $n \in \mathbb{N}$ then assuming exisence of limit, $\lim_{n \longrightarrow \infty} x_n$ is

(a) 1 (b) 4 (c) $\frac{1}{\sqrt{2}}$ (d) does not exist.

(4) Let (x_n) be a monotonic increasing sequence which is not bounded above then (x_n)

(a) is divergent. (b) is convergent. (c) does not exist. (d) none of the above.

(5) $(a_n), (b_n)$ and (c_n) are sequences of real numbers such that $b_n = a_{2n}$ and $c_n = a_{2n+1}$. If (a_n) is convergent then

(a) (b_n) is convergent but (c_n) need not be convergent.
(b) (c_n) is convergent but (b_n) need not be convergent.
(c) both (b_n) and (c_n) are convergent.
(d) none of these.

(6) (a_n) and (b_n) are sequences of real numbers defined as $a_1 = 1$ and $a_{n+1} = a_n + (-1)^n 2^{-n}$ for all $n \geq 1$ and $b_n = \frac{2a_{n+1} - a_n}{a_n}$. Then

(a) (a_n) converges to 0 and (b_n) is Cauchy.
(b) (a_n) converges to a non-zero real number and (b_n) is Cauchy.
(c) (a_n) converges to 0 and (b_n) is not convergent.
(d) (a_n) converges to a non-zero real number and (b_n) is not convergent.

(7) Which of the following is true?

(a) If (x_n) has a convergent subsequence then (x_n) must be a Cauchy sequence.
(b) If (x_n) has a convergent subsequence then (x_n) must be a bounded sequence.
(c) The sequence $\left(\sin \frac{n\pi}{2}\right)$ does not have a convergent subsequence.
(d) The sequence $\left(n \cos \frac{1}{n}\right)$ has a convergent subsequence.

(8) If all subsequences of (x_n) are convergent then (x_n) is

1.4

(a) convergent.
(b) divergent.
(c) not bounded.
(d) none of the above.

(9) If $(x_n), (y_n)$ are Cauchy sequences of real numbers then which of the following statement is not true?

(a) $(x_n + y_n)$ is a Cauchy sequence.
(b) $(x_n * y_n)$ is a Cauchy sequence.
(c) $(c * x_n)$ is a Cauchy sequence.
(d) All the above statements are not true.

(10) Let (x_{n_k}) and (x_{n_r}) be two convergent subsequences of (x_n) converging to same limit then (x_n) is

(a) convergent.
(b) monotone.
(c) divergent.
(d) none of the above.

(11) Which of the following is TRUE?

(a) Every sequence that has a convergent subsequence is a Cauchy sequence.
(b) Every sequence that has a convergent subsequence is a bounded sequence.
(c) The sequence $(\sin n)$ has a convergent subsequence.
(d) The sequence $\big(n \cos(\frac{1}{n})\big)$ has a convergent subsequence.

(12) The sequence (a_n) is defined as follows: $a_1 = 1, a_{n+1} = \dfrac{7a_n + 11}{21}$ for all $n \in \mathbb{N}$. Then (a_n) is

(a) an increasing, divergent sequence.
(b) an increasing sequence with $\lim\limits_{n \longrightarrow \infty} a_n = \dfrac{11}{14}$.
(c) a decreasing sequence which is divergent.
(d) a decreasing sequence with $\lim\limits_{n \longrightarrow \infty} a_n = \dfrac{11}{14}$.

(13) (a_n) and (b_n) are sequences of real numbers such that (a_n) is increasing and (b_n) is decreasing. Under which of the following conditions the sequence $(a_n + b_n)$ is always convergent?

(a) (a_n) and (b_n) are bounded sequences.
(b) (a_n) is bounded above.
(c) (a_n) is bounded above and and (b_n) is bounded below.
(d) $a_n \longrightarrow \infty$ and $b_n \longrightarrow -\infty$.

(B) Descriptive

(1) Using $\epsilon - n_0$ definition, show that the following sequences are Cauchy.

(a) $x_n = \frac{2}{n^2}$ for $n \in \mathbb{N}$.
(b) $x_n = \frac{(-1)^n}{n^2}$ for $n \in \mathbb{N}$.
(c) $x_n = \frac{n+2}{n+1}$ for $n \in \mathbb{N}$.
(d) $x_n = \frac{n+3}{2n+1}$ for $n \in \mathbb{N}$.
(e) $x_n = \frac{1}{3^n}$ for $n \in \mathbb{N}$.

(2) State whether the following statements are true pr false with justification.

(a) If $(x_n + y_n)$ is a Cauchy sequence of real numbers then either (x_n) or (y_n) is Cauchy.
(b) If $(x_n * y_n)$ is a Cauchy sequence of real numbers then either (x_n) or (y_n) is Cauchy.
(c) If (x_n^2) is a Cauchy sequence of real numbers then (x_n) is Cauchy.

(3) Prove the following.

(a) (x_n) and (y_n) are Cauchy sequences of real numbers. If $a_n = |x_n - y_n|$ for all $n \in \mathbb{N}$ then (a_n) is also Cauchy.
(b) If (x_n) is a Cauchy sequence of integers then there exists $N \in \mathbb{N}$ such that $x_n = c$ for all $n \geq N$ where c is an integer constant.

(4) Check whether the following sequences are monotonic and bounded.

(i) $x_n = \frac{5}{n+1}$ for $n \in \mathbb{N}$.
(ii) $x_n = \frac{n}{n+2}$ for $n \in \mathbb{N}$.
(iii) $x_n = \frac{n^2}{n+2}$ for $n \in \mathbb{N}$.
(iv) $x_n = \frac{n}{n^2+1}$ for $n \in \mathbb{N}$.
(v) $x_n = \frac{2^n 3^n}{5^{n+1}}$ for $n \in \mathbb{N}$.
(vi) $x_n = n^3 - n$ for $n \in \mathbb{N}$.
(vii) $x_n = \frac{n+1}{n-1}$ for $n \in \mathbb{N}$.
(viii) $x_n = \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(2n-1)^2}$ for all $n \in \mathbb{N}$.
(ix) $x_n = \frac{1}{1*3} + \frac{1}{2*3^2} + \cdots + \frac{1}{n*3^n}$ for all $n \in \mathbb{N}$.
(x) $x_n = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$ for all $n \in \mathbb{N}$.

(5) $x_1 = 1$ and $x_{n+1} = \frac{3x_n + 2}{6}$, for all $n \in \mathbb{N}$. Prove that (x_n) is monotonically decreasing and bounded below.

(6) Show that (x_n) is monotonic in the following examples. Also find an upper bound if it is monotonically increasing and a lower bound if it is monotonically decreasing.

(i) $x_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$ for all $n \in \mathbb{N}$.
(ii) $x_n = \frac{1}{1^2+1} + \frac{1}{2^2+1} + \cdots + \frac{1}{n^2+1}$ for all $n \in \mathbb{N}$.
(iii) $x_n = \frac{1}{1*2} + \frac{1}{2*3} + \cdots + \frac{1}{n*(n+1)}$ for all $n \in \mathbb{N}$.
(iv) $x_n = \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1}$ for all $n \in \mathbb{N}$.
(v) $x_n = \frac{n+1}{n-1}$, for all $n \in \mathbb{N}$.

1.4

(vi) $x_n = \sum_{k=0}^{n} \frac{1}{k!}$ (Hint: $2^{k-1} \leq k!$ for all $k \in \mathbb{N}$).

(7) Prove that (x_n) is not convergent in the following examples by showing that it has two convergent subsequences converging to different limits.

(i) $x_n = (-1)^n + \frac{1}{n}$ for $n \in \mathbb{N}$.

(ii) $x_n = \sin\left(\frac{n\pi}{2}\right)$ for $n \in \mathbb{N}$.

(iii) $x_n = \cos\left(\frac{n\pi}{3}\right)$ for $n \in \mathbb{N}$.

(8) Give an example of each of the following.

(i) a sequence of real numbers such that no subsequence is convergent.

(ii) an unbounded sequence that has a convergent subsequence.

1.5 Practical 1.5: Differential Equations

1.5.1 Prerequisite of Practical 1.5

(1) **Differential Equation:** An equation involving one dependent variable and its derivatives with respect to one or more independent variables is called a differential equation. For example $m\frac{d^2y}{dx^2} = -ky$. Here y is the dependent variable and x is an independent variable.

(2) **Ordinary differential equation**: An ordinary differential equation is one in which there is only one independent variable, so that all the derivatives occurring in it are ordinary derivatives.

(3) **Order**: The order of a differential equation is the order of the highest derivative present in the equation.

(4) **degree**: The degree of a differential equation is the power of the highest-order derivative.

(5) **General Ordinary differential equation** of the n^{th} order is $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0$, or using the notation for derivatives, $F(x, y, y', y'', \cdots y^{(n)}) = 0$.

(6) **Linear ordinary differential equation**: The linear ordinary differential equation is the ordinary differential equation in which the dependent variable and its derivatives occur in the first degree and the equation does not contain their product.
The general first order linear equation is $\frac{dy}{dx} + p(x)y = q(x)$.

(7) **Partial differential equation**: A partial differential equation is the one involving more than one independent variable, so that the derivatives occurring in it are partial derivatives.

(8) A function $f(x, y)$ is called homogeneous of degree n if $f(tx, ty) = t^n\, f(x, y)$ for all suitably restricted x, y and t.
e.g. $x^2 + xy$, $\sqrt{x^2 + y^2}$, $\sin\left(\frac{x}{y}\right)$ are homogeneous of degrees $2, 1$ and 0.
The differential equation $M(x, y)dx + N(x, y)dy = 0$ is said to be **homogeneous** if the functions M and N are homogeneous functions of the same degree.

(9) **Exact differential equation**:
Consider the differential equation $M(x, y)dx + N(x, y)dy = 0$.
If there exists a function $f(x, y)$ having continuous partial derivatives such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$ then the given differential equation is said to be an exact differential equation of first order and first degree.

(10) The necessary and sufficient condition for $M(x, y)dx + N(x, y)dy = 0$ to be an exact differential equation is $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

(11) If the differential equation $M(x, y)dx + N(x, y)dy = 0$ is non-exact but there exists a function $\mu(x, y)$ such that the equation $\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0$. is exact then the function $\mu(x, y)$ is called an **integrating factor** of $M(x, y)dx + N(x, y)dy = 0$.

(12) **Rules for finding integrating factors.**
RULES I and II
RULE I. Suppose the differential equation $M(x, y)dx + N(x, y)dy = 0$ is non-exact and **homogeneous**. If $M\ x + N\ y \neq 0$, then $\dfrac{1}{M\ x + N\ y}$ is an integrating factor of $Mdx + Ndy = 0$
RULE II. Suppose the differential equation $M(x, y)dx + N(x, y)dy = 0$ is non-exact. If $M\ x - N\ y \neq 0$ and the equation has the form $f_1(xy)\ y\ dx + f_2(xy)\ x\ dy = 0$, then $\dfrac{1}{M\ x - N\ y}$ is an integrating factor.
Note:

(1) $M\ x + N\ y = 0 \Longrightarrow \dfrac{M}{N} = \dfrac{-y}{x}$
$\therefore$ the differential equation becomes, $\dfrac{dy}{dx} = \dfrac{-M}{N} = \dfrac{y}{x}$
$\therefore \dfrac{dy}{dx} = \dfrac{y}{x}$
$\therefore \dfrac{dy}{y} = \dfrac{dx}{x}$
Integrating both sides we get,
$\log y = \log x + \log c$
$\therefore \dfrac{y}{x} = c$ is a solution.

(2) When $M\ x - N\ y = 0 \Longrightarrow \dfrac{M}{N} = \dfrac{y}{x}$
$\therefore$ the differential equation becomes, $\dfrac{dy}{dx} = \dfrac{-M}{N} = \dfrac{-y}{x}$
$\therefore \dfrac{dy}{dx} = \dfrac{-y}{x}$

$\therefore \dfrac{dy}{y} = -\dfrac{dx}{x}$
Integrating both sides we get,
$\log y = -\log x + \log c$
$\therefore xy = c$ is a solution.

RULE III. Suppose the differential equation $M(x,y)dx + N(x,y)dy = 0$ is non-exact. If $\dfrac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N}$ is a function of x alone, say $f(x)$ then $e^{\int f(x)\ dx}$ is an integrating factor.

RULE IV Suppose the differential equation $M(x,y)dx + N(x,y)dy = 0$ is non-exact. If $\dfrac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M}$ is a function of y alone, say $F(y)$ then $e^{\int F(y)\ dy}$ is an integrating factor.

(13) The general first order linear equation is $\dfrac{dy}{dx} + p(x)y = q(x)$.
The integrating factor of the first order linear equation $\dfrac{dy}{dx} + p(x)y = q(x)$ is $e^{\int p(x)\ dx}$ and its solution is $y\ e^{\int p(x)dx} = \int e^{\int p(x)dx} q(x)\ dx + c$.

(14) **Equations reducible to the linear form : Bernoulli's Differential Equations**
Sometimes the equations that are not linear can be reduced to the linear form. In particular, this is the case with equations of the form $\dfrac{dy}{dx} + P(x)\ y = Q(x)\ y^n$.
Dividing by y^n and multiplying by $(-n+1)$, this equation becomes
$(-n+1)y^{-n}\dfrac{dy}{dx} + (-n+1)\ P(x)\ y^{-n+1} = (-n+1)\ Q(x)y^n$
put $v = y^{-n+1}$ $\quad \therefore dv = (-n+1)y^{-n}dy$ and the equation (1) becomes,
$\dfrac{dv}{dx} + (-n+1)P(x)\ v = (-n+1)Q(x)$
i.e. $\dfrac{dv}{dx} + (1-n)P(x)\ v = (1-n)\ Q(x)$
which is linear in v.
Hence the equation of the form $\dfrac{dy}{dx} + P(x)\ y = Q(x)\ y^n$ can be reduced to the first order linear ordinary differential equation.

1.5.2 PRACTICAL 1.5

(A) Objective Questions

Choose correct alternative in each of the following:

(1) The degree of the O.D.E. $y' + x = (y - xy')^{-2}$ is

(a) 3 (b) $\frac{1}{2}$ (c) 2. (d) 1.

(2) $y^2 = cx$ is the general solution of which of the following first order O.D.E.?

(a) $\frac{dy}{dx} = \frac{x}{2y}$. (b) $\frac{dy}{dx} = \frac{y}{2x}$. (c) $\frac{dy}{dx} = \frac{2y}{x}$. (d) $\frac{dy}{dx} = \frac{2x}{y}$.

(3) The following is an I.F. of the Linear first order O.D.E. $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x only

(a) e^{-Pdx} (b) e^{Qdx} (c) e^{Pdx} (d) e^{-Qdx}

(4) A necessary and sufficient condition for a first order O.D.E. $M(x,y)dx + N(x,y)dy = 0$ to be exact is

(a) $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ (b) $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (c) $\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$ (d) $\frac{\partial M}{\partial x} \neq \frac{\partial N}{\partial y}$

(5) A differential equation is considered to be ordinary if it has

a) more than one dependent variable.
b) one independent variable.
c) more than one independent variable.
d) None of these.

(6) Integrating factor of the first order differential equation $(x^2 + y^2 + x)dx + xydy = 0$ is

(a) y. (b) x. (c) x^{-1}. (d) y^{-1}.

(7) Which of the following is not an exact first order differential equation?

(a) $(2x + y - 4)dx + (x - 2y + 7)dy = 0$
(b) $2xdy + 2y^2dx = 0$
(c) $(x^3 - 6xy - 4y^2)dx + (2y^2 - 6xy - 4x^2)dy = 0$
(d) None of the above.

(8) If the general solution of a differential equation is $(y+a)^2 = bx$ where a and b are arbitrary constants then the order of the differential equation is

(a) 1 (b) 2 (c) 3 (d) None of these.

(9) The following is an I.F. of the first order O.D.E. $(x + y^3)dx + 6xy^2dy = 0$

(a) $x^{-\frac{1}{2}}$ (b) y^{-3} (c) $y^{-\frac{1}{2}}$. (d) None of these.

(10) The equation of the orthogonal trajectories to the family of parabolas $y^2 = 2x + c$ is (Here c and k are arbitrary constants).

(a) $y = ke^{-2x}$. (b) $y = ke^{-x}$. (c) $y = ke^{x}$. (d) None of these.

(11) The degree of the O.D.E. $\frac{dy}{dx} = \sqrt{y^3 - 2x\frac{dy}{dx} + 4}$ is

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 3.

(12) Which of the following is an exact first order O.D.E.

(a) $(x^2 - 2x + 2y^2)dx + 2xydy = 0$
(b) $2xydx + (x^2 + 2)dy = 0$
(c) $x^2ydy - ydx = 0$
(d) None of these.

(B) Descriptive Questions

(1) Test each of the following equations for exactness, and solve it if it is exact:

(i) $e^y dx + (xe^y + 2y)dy = 0$
(ii) $(2xy^3 + y\cos x)\ dx + (3x^2y^2 + \sin x)\ dy = 0.$
(iii) $\left(x + \frac{2}{y}\right) dy + ydx = 0$

(2) Determine which of the following equations are exact, and solve the ones that are exact.

(i) $\left(x + \frac{2}{y}\right)\ dy + y\ dx = 0$
(ii) $(\sin x \tan y + 1)\ dx + \cos x \sec^2 y\ dy = 0$
(iii) $(y - x^3)\ dx + (x + y^3)\ dy = 0$
(iv) $(2y^2 - 4x + 5)\ dx = (4 - 2y + 4xy)\ dy$
(v) $(y + y\cos xy)\ dx + (x + x\cos xy)\ dy = 0$
(vi) $\cos x \cos^2 y\ dx + 2\sin x \sin y \cos y\ dy = 0$
(vii) $(\sin x \sin y - xe^y)dy = (e^y + \cos x \cos y)\ dx$
(viii) $-\frac{1}{y}\sin\left(\frac{x}{y}\right)\ dx + \frac{x}{y^2}\sin\left(\frac{x}{y}\right)\ dy = 0$
(ix) $(1 + y)\ dx + (1 - x)\ dy = 0$
(x) $(2xy^3 + y\cos x)\ dx + (3x^2y^2 + \sin x)\ dy = 0$
(xi) $dx = \frac{y}{1 - x^2y2}\ dx + \frac{x}{1 - x^2y^2}dy$

(3) Solve the following differential equations:

(i) $(x^2y - 2xy^2)\ dx - (x^3 - 3x^2y)\ dy = 0$
(ii) $y - x\frac{dy}{dx} = x + y\frac{dy}{dx}$
(iii) $(x^4 + y^4)\ dx - xy^3\ dy = 0$
(iv) $y^2\ dx + (x^2 - xy - y^2)\ dy = 0)$.
(v) $(2x - y)e^{\frac{y}{x}}\ dx + (y + xe^{\frac{y}{x}})\ dy$

(4) Solve the following differential equations:

(i) $y(xy + 2x^2y^2)\ dx + x(xy - x^2y^2)\ dy$

(ii) $y(x^2y^2 + 2)\ dx + x(2 - 2y^2x^2)\ dy = 0$

(iii) $y(2xy+1)\ dx + x(1+2xy-x^3y^3)\ dy = 0.$

(iv) $y(xy - 3)\ dx + x(3xy - 3)\ dy = 0.$

(v) $y(x^2y^2 - 3xy - 3)\ dx + x(x^2y^2 - 2xy - 3)\ dy = 0$

(5) (i) $(x^2 + y^2 + 2x)\ dx + (2y)dy = 0$ (ii) $(x^2 + y^2)\ dx - 2xy\ dy = 0$

(6) Solve the following differential equations:

(i) $(3x^2y^4 + 2xy)\ dx + (2x^3y^3 - x^2)dy = 0.$

(ii) $(y^4 + 2y)\ dx + (xy^3 + 2y^4 - 4x)dy = 0$

(iii) $(x^4 + y^4)dx - xy^3dy = 0$

(7) Solve the following linear differential equations:

(i) $x\dfrac{dy}{dx} - ay = x + 1$

(ii) $\dfrac{dy}{dx} + y = e^{-x}$

(iii) $\cos^2 x\ \dfrac{dy}{dx} + y = \tan x$

(iv) $(1 + y^2)\ dx = (\tan^{-1} y - x)\ dy$

(v) $(x + 1)\ \dfrac{dy}{dx} - ny = e^x\ (x + 1)^{n+1}$

(vi) $(x^2 + 1)\dfrac{dy}{dx} + 2xy = 4x^2$

(8) Solve the following reducible to linear differential equations:

(i) $\dfrac{dy}{dx} + \dfrac{1}{x}\ y = x^2\ y^6$

(ii) $\dfrac{dy}{dx} + \dfrac{2}{x}\ y = 3x^2\ y^{\frac{4}{3}}$

(iii) $\dfrac{dy}{dx} + \dfrac{xy}{1 - x^2} = xy^{\frac{1}{2}}$

(iv) $3x\ (1 - x^2)\ y^2\ \dfrac{dy}{dx} + (2x^2 - 1)\ y^3 = ax^3$

(v) $3y^2\dfrac{dy}{dx} + 2xy^3 = 4xe^{-x^2}.$

(9) Find the particular solution of the following differential equations.

(i) $y' - y = e^x, y(1) = 0$

(ii) $y' + y = (1 + x)^2, y(0) = 0$

(iii) $y' - x^3y = -4x^3, y(0) = 6$

(iv) $y' - y\cot x = 2x - x^2\cot x, y\left(\frac{\pi}{2}\right) = \dfrac{\pi^2}{4} + 1$

(v) $y' - \left(1 + \dfrac{3}{x}\right) y = x + 2, y(1) = e - 1$

(vi) $y' + y\tan x = 2x\cos x, y(0) = -1$

(vii) $xy' = (1 + x)y, y(2) = 6e^2.$

(10) Solve the following differential equations that are linear in x.

(i) $\dfrac{dx}{dy} = e^{-3y} - 3x$

(ii) $\dfrac{dx}{dy} = 2\sin y - x$

(iii) $\dfrac{dx}{dy} = 2y(1 - 2y + 2y^{-1}x)$

(iv) $\dfrac{dx}{dy} = e^y + x$

(v) $y\dfrac{dx}{dy} = 2x + y^3e^y$

1.5

1.6 Practical 1.6: Applications of Differential Equations, Equations reduced to first order equations

1.6.1 Prerequisite Practical 1.6

(1) **Orthogonal Trajectory**: An orthogonal trajectory of a family of curves is a curve that cuts all the members of the family at right angle.

(2) The simplest mathematical model of **population growth** is obtained by assuming that the rate of increase of the population at any time is proportional to the size of the population at that time. If we let $P(t)$ denote the population at time t, then
$$\frac{dP}{dt} = kP$$
where k is a positive constant. Separating the variables and integrating we get
$$P = P_0\ e^{kt}$$
where P_0 denotes the population at $t = 0$ and P is a population at time t. We denote it by $P(t)$
Thus $\quad P(t) = P_0\ e^{kt}$
This law predicts an exponential increase in the population with time and gives a reasonably accurate description of the growth of certain algae, bacteria, and cell cultures. The time taken for such a culture to double in size is called the **doubling time.** This is time t_d when $P = 2P_0$.
Substituting in the above equation, we get
$$2P_0 = P_0 e^{kt_d}$$
Dividing both sides by P_0 and taking logarithms, we fine
$$kt_d = \log 2.$$
So that the doubling time is $t_d = \frac{1}{k} \log 2$.

(3) In a circuit, the voltage is denoted by $V(t)$ and the current is denoted by $i(t)$ where t is the time.
The differential equation is $\quad L * \frac{di}{dt} + R * i = V(t). \quad$ where L is the coefficient of induction and R the resistance of the circuit. $(R > 0, L > 0)$

(4) Newton's Law of Cooling: The rate of heat loss of a body is directly proportional to the difference between the temperature of body itself and temperature of the surrounding medium. The equation is $\frac{dT}{dt} = -k(T - T_0)$.

(5) **Second-order differential equations reducible to the first order.**

(i) If in a second-order equation the dependent variable y does not appear explicitly, the equation is of the form $F(x, y', y'') = 0$. The substitution $y' = z$ reduces the given equation into a first-order differential equation in z and from its solution, the solution of the original equation can be obtained. For example, we will reduce $2xy'' = 3y'$ to the first order and solve.
Put $z = \frac{dy}{dx}$. This implies $\frac{dz}{dx} = \frac{d^2y}{dx^2}$.

Hence the given equation reduces to $2x\dfrac{dz}{dx} = 3z$. That is,

$$
\begin{aligned}
\frac{dz}{z} &= \frac{3}{2}\frac{dx}{x}. \\
\ln z &= \frac{3}{2}\ln x + \ln c \\
z &= x^{\frac{3}{2}}c \\
\frac{dy}{dx} &= x^{\frac{3}{2}}c \\
dy &= x^{\frac{3}{2}}cdx \\
y &= \frac{2}{5}x^{\frac{5}{2}}c + K
\end{aligned}
$$

So, we will write the solution as $y = c_1 x^{\frac{5}{2}} + c_2$.

(ii) Another type of equations reducible to first order is $F(y, y', y'') = 0$, in which the independent variable x does not appear explicitly. We substitute $y' = z$. Differentiate this again, we get, $\dfrac{d^2y}{dx^2} = \dfrac{dz}{dx} = \dfrac{dz}{dy}\dfrac{dy}{dx} = \dfrac{dz}{dy}z = z\dfrac{dz}{dy}$. For example, we will solve $y\ y'' + y'^2 = 0$. (we assume $z \neq 0$ for all x.)

$$
\begin{aligned}
yz\frac{dz}{dy} + z^2 &= 0 \\
y\frac{dz}{dy} &- z \\
\frac{dz}{z} &- \frac{dy}{y} \\
\ln z &= -\ln y + \ln c \\
zy &= c \\
\frac{dy}{dx}y &= c \\
ydy &= cdx \\
\frac{y^2}{2} &= cx + k \\
y^2 &= 2cx + 2k
\end{aligned}
$$

The solution is $y^2 = c_1 x + c_2$.

xxxxxxxxxxxx

1.6.2 PRACTICAL 1.6

(A) Objective Questions

Choose correct alternative in each of the following:

(1) The equation of the orthogonal trajectories to the family of parabolas $y^2 = 2x + c$ is (Here c and k are arbitrary constants).

(a) $y = ke^{-2x}$. (b) $y = ke^{-x}$. (c) $y = ke^{x}$. (d) None of these.

(2) The number of bacteria in a certain culture grows at a rate that is proportional to the number present. If the number increased from 500 to 2000 in 2 hr then the doubling time is

(a) 1 hr. (b) 2 hrs. (c) 1/2 hr. (d) None of these.

(3) The equation of the orthogonal trajectories to the family of $y = c(\sec x + \tan x)$ is (Here c and k are arbitrary constants).

(a) $y^2 + 2\sin x = k$. (b) $y^2 - 2\sin x = k$. (c) $y^2 - 2\cos x = k$. (d) None of these.

(4) The equation of the orthogonal trajectories to the family of $x - 4y = c$ is (Here c and k are arbitrary constants).

(a) $y + 4x = k$. (b) $4y + 3x = k$. (c) $y + x = k$. (d) None of these.

(5) The equation of the orthogonal trajectories to the family of $x = ce^{y^2}$ is (Here c and k are arbitrary constants).

(a) $e^{-x^2} = ky$. (b) $e^{-x^2} = ky^2$. (c) $e^{x} = ky$. (d) None of these.

(6) A certain bacteria grows at a rate that is proportional to the number present. Let P_0 be the initial population present.

If it is found that the number doubles in 4 hours, then the population at the end of 12 hours is

(a) $8 * P_0$. (b) $7 * P_0$. (c) $9 * P_0$. (d) None of these.

(7) A copper ball is heated to a temperature of $100°C$. Then at time $t = 0$ it is placed in water that is maintained at a temperature of $30°C$. At the end of 3 min. the temperature of the ball is reduced to $70°C$. The time at which the temperature of the ball is reduced to $31°C$ is

(a) approximately 23 minutes.
(c) approximately 25 minutes.
(b) approximately 20 minutes.
(d) None of these.

(8) The initial concentration of a certain medication in human blood stream is $12mg/cm^3$. If every hour the concentration is reduced by 14%, then the amount of medication in the bloodstream (A), in mg/cm^3, over time (t), in hours is

(a) $12(0.86)^t$. (b) $10(0.86)^t$. (c) $14(0.68)^t$. (d) None of these.

(9) An RLcircuit has an emf given by $3\sin 2t$, a resistance of 10 Ω , an inductance of 0.5H, and an initial current of 6 A. The current in the circuit at any time t is

(a) $10i = 1 - e^{-50t}$. (b) $i = 1 - e^{-50t}$. (c) $10i = 1 - e^{50t}$. (d) None of these.

(10) If we reduce $y'' = y'$ to the first order differential equation then its solution is

(a) $y = c_1e^x + c_2$. (b) $y = c_1e^{-x} + c_2$. (c) $y = c_1e^{x^2} + c_2$. (d) None of these.

(11) If we reduce $yy' = 2y'^2$ to the first order differential equation then its solution is

(a) $y = (c_1x + c_2)^{-1}$. (b) $y = (c_1x^2 + c_2x)^{-1}$. (c) $y = (c_1x + c_2)$. (d) None of these.

(B) Descriptive Questions

(1) Find the orthogonal trajectories of each of the family of curves:

(i) $x^2 + y^2 - 2cx = 0$
(ii) $y^2 = cx^3$
(iii) $x - 4y = c$
(iv) $x^2 + y^2 = c$
(v) $x^2 - y^2 = c$
(vi) $y^2 = cx^3$
(vii) $e^x + e^{-y} = c$
(viii) $y = c(\sec x + \tan x)$
(ix) $x^3 = 3(y - c)$
(x) $x = ce^{y^2}$
(xi) $y = ce^{-mx}$ m fixed
(xii) $x^2 - y^2 = cx$

(2) The number of bacteria in a certain culture grows at a rate that is proportional to the number present. If the number increased from 500 to 2000 in 2 hr, determine the number present after 12 hrs and find the doubling time. (Assume that when $t = 0, P_0 = 500$)

(3) A certain population of bacterial is known to grow at a rate proportional to the amount present in a culture that provides plentiful food and space. Initially there are 250 bacterial, and after seven hours 800 bacteria are observed in the culture. Find an expression for the approximate number of bacteria present in the culture at any time t. Also determine the approximate number of bacteria that will be present in the culture described in the above example after 24 hours and Determine the amount of time it will take for the bacteria described in the above example to increase to 2500.

(4) The number of bacteria in a certain culture grows at a rate that is proportional to the number present. Find an expression for the approximate number of bacteria in such a culture if the initial number is 300 and if it is observed that the population has increased by 20 percent after 2 hours. Also determine the number of bacteria that will be present in the culture after 24 hours as well as after after one week. Further, determine the amount of time it will take the culture to double its original population.

(5) The initial concentration of a certain medication in human blood stream is $12mg/cm^3$. If every hour the concentration is reduced by 14%, then the find the amount of medication in the bloodstream (A), in mg/cm^3, over time (t), in hours.
(Hint: The differential equation is $\dfrac{dA}{dt} = kA$ for some constant of proportion k.)

(6) A certain bacteria grows at a rate that is proportional to the number present.

(i) If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours.

(ii) If there are 10^4 bacteria at the end of 3 hours and 4×10^4 at the end of 5 hours, how many were there initially?

(7) In a culture of yeast the amount of active ferment grows at a rate proportional to the amount present. If the amount doubles in 1 hour, how many times the original amount may be anticipated at the end of $\dfrac{11}{4}$ hours?

(8) The population of certain country is known to increase at a rate proportional to the number of people presently living in the country. If after 2 years the population has doubled, and after 3 years the population is 20000, find the number of people initially leaving in the country.

(9) Bacteria are placed in a nutrient solution and allowed to multiply. Food is plentiful but space is limited, so competition for space will force the bacteria population to stabilize at some constant level M. Determine an expression for the population at time t if the growth rate of the bacteria is jointly proportional to the number of bacteria present and the difference between M and the current population.
(Hint: The differential equation is $\dfrac{dP}{dt} = kP(M - P)$.)

(10) If the population of a country doubles in 20 years, in how many years will it triple under the assumption that the rate of increase is proportional to the number of inhabitants?

(11) Reduce to the first order and solve:

(i) $2xy'' = 3y'$

(ii) $y'' = y'$

(iii) $y'' + y' = x + 1$

(iv) $y'' = 1 + y'^2$

(v) $xy'' + y' = y'^2$

(12) Reduce to the first order and solve:

(i) $yy'' = 2y'^2$

(ii) $yy'' + y'^2 = 0$

(iii) $y'' + e^y y'^3 = 0$

(iv) $y'' + 2y'^2 = 0$

(v) $y'' + y'^3 \cos y = 0$

(vi) $y'' + (1 + y^{-1})y'^2 = 0$

(13) A particle moves on a straight line so that its acceleration is equal to three times its velocity. At $t = 0$ its displacement from the origin is $1m$ and its velocity is $1.5\ m/s$. Find the time when the displacement is $10m$.

1.7 Practical 1.7: Miscellaneous Theory Questions

1.7.1 Practical 1.7: Miscellaneous Theory Questions on UNIT 1

(1) Prove the following properties of real numbers.

(i) Additive identity is unique.
(ii) Multiplicative identity is unique.
(iii) Every real number has unique additive inverse.
(iv) Every non-zero real number has unique multiplicative inverse.

(2) Let $a, b \in \mathbb{R}$. Then

(i) there is exactly one $x \in \mathbb{R}$ such that $a + x = b$.
(ii) if $a \neq 0$, there is exactly one $x \in \mathbb{R}$ such that $ax = b$.

(3) If $a, b, c \in \mathbb{R}$, then using only the algebraic properties, prove the following.

(i) $-(-a) = a$
(ii) $(a^{-1})^{-1} = a$
(iii) $a0 = 0 = 0a$
(iv) $(-a)b = -(ab) = a(-b)$
(v) $(-a)(-b) = ab$
(vi) $a(b - c) = ab - ac$
(vii) If $ab = 0$ then $a = 0$ or $b = 0$
(viii) If $a \neq 0, b \neq 0$, then $(ab)^{-1} = b^{-1}a^{-1}$
(ix) If $a \neq 0$, then $(-a)^{-1} = -a^{-1}$
(x) $-0 = 0$
(xi) $1^{-1} = 1$

(4) x, y, z are real numbers, then prove the following.

(i) $x < 0 \implies -x > 0$
(ii) $x < y$ and $y < z \implies x < z$
(iii) $x < y \implies x + z < y + z$
(iv) $x < y$ and $z > 0 \implies xz < yz$
(v) $x < y$ and $z < 0 \implies xz > yz$
(vi) $x > 0, y < 0 \implies xy < 0$
(vii) $1 > 0$.
(viii) If $x > 0$, then $x^{-1} > 0$ and if $x < 0$ then $x^{-1} < 0$.
(ix) If $0 < x < y$, then $0 < y^{-1} < x^{-1}$.
(x) If $x < y < 0$, then $y^{-1} < x^{-1} < 0$.

(5) Define absolute value of a real number and prove the following.
For $x, y \in \mathbb{R}$, the following properties hold.

(i) $|x| \geq 0$.
(ii) $|x| = 0 \iff x = 0$
(iii) $|x| = |-x|$.
(iv) $|x| = \max\{x, -x\}$.
(v) $-|x| \leq x \leq |x|$.
(vi) $|xy| = |x||y|$.
(vii) If $y \neq 0$ then $\left|\frac{1}{y}\right| = \frac{1}{|y|}$.
(viii) If $y \neq 0$ then $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$.
(ix) If $r \in \mathbb{R}, r > 0$, then $|x| \leq r$ if and only if $-r \leq x \leq r$.
(x) $|x + y| \leq |x| + |y|$.
(xi) $|x - y| \leq |x| + |y|$.
(xii) $|x - y| \geq \big||x| - |y|\big|$.

(6) State and prove AM-GM Inequality for $a, b \in \mathbb{R}^+$.

(7) If $a_1, a_2, \cdots, a_n$ are non-negative real numbers, then prove that $\dfrac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots$

(8) State and prove Cauchy Schwartz Inequality for $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ any real numbers.
If $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are any real numbers then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \left(\sum_{k=1}^{n} {a_k}^2\right)\left(\sum_{k=1}^{n} {b_k}^2\right).$$

(9) State and prove Hausdorff Property.

(10) Define a bounded set in $\mathbb{R}$, lower bound and upper bound of a subset of $\mathbb{R}$, infimum (g.l.b.) and supremum (l.u.b.)of a subset of $\mathbb{R}$ and prove that a nonempty set $S \subseteq \mathbb{R}$ is bounded if and only if there is $M \in \mathbb{R}^+$ such that, $|x| \leq M \ \ \forall\, x \in S$.

(11) State L.U.B. Axiom. Prove that every nonempty set S of real numbers that is bounded below has an infimum in $\mathbb{R}$.

(12) Prove the following:

(i) If a non-empty subset of $\mathbb{R}$ has a supremum then it is unique.

(ii) If a non-empty subset of $\mathbb{R}$ has an infimum then it is unique.

(13) Let S be a nonempty subset of $\mathbb{R}$ such that S is bounded above. Let $M \in \mathbb{R}$. Then prove that $M = \sup S$ if and only if
(I) M is an upper bound of S.
(II) for any $\varepsilon > 0$, there is an element $a \in S$ such that $M - \varepsilon < a \leq M$.

(14) Let S be a nonempty subset of $\mathbb{R}$ such that S bounded below. Let $m \in \mathbb{R}$.
Then prove that $m = \inf S$ if and only if
(I) m is a lower bound of S.
(II) for any $\varepsilon > 0$, there is an element $a \in S$ such that $m \leq a < m + \varepsilon$.

(15) Prove the following.

(i) S is a non-empty subset of $\mathbb{R}$. If S is bounded above then the set of all upper bounds of S is bounded below.

(ii) S is a non-empty subset of $\mathbb{R}$. If S is bounded below then the set of all lower bounds of S is bounded above.

(16) Let $S \subseteq \mathbb{R}$ be non-empty and let $\alpha \in \mathbb{R}$. Prove that $\alpha = \sup S$ if and only if for every $n \in \mathbb{N}$, the number $\alpha - \dfrac{1}{n}$ is not an upper bound of S but $\alpha + \dfrac{1}{n}$ is an upper bound of S.

(17) Let A, B be nonempty subsets of $\mathbb{R}$.

(i) If $A \subseteq B$ and B is bounded then prove that A is also bounded.

(ii) If $A \subseteq B$, and B is bounded, then prove that $\inf B \leq \inf A \leq \sup A \leq \sup B$.

(iii) If A and B are bounded then:

(a) $A \cup B$ and $A \cap B$ are also bounded.
(b) $\inf(A \cup B) = \min\{\inf A,\ \inf B\}$.
(c) $\sup(A \cup B) = \max\{\sup A,\ \sup B\}$.
(d) $\inf(A \cap B) \geq \max\{\inf A,\ \inf B\}$.
(e) $\sup(A \cap B) \leq \min\{\sup A,\ \sup B\}$.

(14) State and Prove Archimedean Property. (Given any $x \in \mathbb{R}$, there is $n \in \mathbb{N}$ such that $n > x$.)

(15) Show that there is no rational number r such that $r^2 = p$; p is a prime.

(16) State and prove Density theorem. (If x and y are any real numbers with $x < y$, then there exists a rational number r such that $x < r < y$. Moreover this number r can always be selected so that it is nonzero.)

(17) If x and y are any real numbers with $x < y$, then there exists an irrational number s such that $x < s < y$.

1.7.2 Practical 1.7: Miscellaneous Theory Questions on UNIT II

(1) Define a bounded sequence, monotonic increasing sequence, monotonic decreasing sequence, monotonic sequence.

(2) Define a convergent sequence. Prove that every convergent sequence converges to a unique limit.

(3) Prove that every convergent sequence is bounded.

(4) Algebra of Convergent Sequences: Let $x_n \longrightarrow p$ and $y_n \longrightarrow q$. Then using $\epsilon - n_0$ definition prove the following.

(i) $x_n + y_n \longrightarrow p + q$,

(ii) $x_n - y_n \longrightarrow p - q$,

(iii) $rx_n \longrightarrow rp$ for any $r \in \mathbb{R}$,

(iv) $x_n y_n \longrightarrow pq$,

(v) $|x_n| \longrightarrow |p|$,

(vi) If $x_n \neq 0 \ \forall \ n \in \mathbb{N}$ and $p \neq 0$ then $\dfrac{1}{x_n} \longrightarrow \dfrac{1}{p}$.

(vii) If $x_n \longrightarrow p$ and $y_n \longrightarrow q, y_n \neq 0 \ \forall \ n \in \mathbb{N}$ and $q \neq 0$ then $\dfrac{x_n}{y_n} \longrightarrow \dfrac{p}{q}$.

(ix) If there is $n_0 \in \mathbb{N}$ such that $x_n \leq y_n$ for all $n \geq n_0$, then $p \leq q$.

(5) State and prove Sandwich Theorem for sequences.(Let $(x_n), (y_n)$ and (z_n) be sequences and $p \in \mathbb{R}$ be such that $x_n \leq z_n \leq y_n$ for all $n \in \mathbb{N}$ and $x_n \longrightarrow p$ as well as $y_n \longrightarrow p$. Then $z_n \longrightarrow p$.)

(6) Prove the Convergence of the following standard sequences:

(1) $\lim\limits_{n \longrightarrow \infty} \dfrac{1}{1 + na} = 0 \quad \forall \ a > 0.$

(2) $\lim\limits_{n \longrightarrow \infty} b^n = 0 \quad \forall \ b, \ |b| < 1.$

(3) $\lim\limits_{n \longrightarrow \infty} c^{\frac{1}{n}} = 1, \quad \forall \ c > 0.$

(4) $\lim\limits_{n \longrightarrow \infty} n^{\frac{1}{n}} = 1$

(7) Define a monotonic sequence. Prove the following.

(i) If a monotonically increasing sequence is bounded above then it is convergent and it converges to the supremum of $\{x_n : n \in \mathbb{N}\}$.

(ii) If a monotonically decreasing sequence is bounded below then it is convergent and it converges to the infimum of $\{x_n : n \in \mathbb{N}\}$.

(iii) A monotonic sequence is convergent if and only if it is bounded.

(8) prove that the sequence $\left(1+\frac{1}{n}\right)^n$ is convergent.

(9) Define a Cauchy sequence. Prove that every Cauchy sequence is bounded.(converse not true).

(10) Every convergent sequence is Cauchy.

(11) Define a subsequence of a sequence. Prove that a subsequence of a convergent sequence is convergent and it converges to the same limit as that of the sequence.

1.7.3 Practical 1.7: Miscellaneous Theory Questions on UNIT III

(1) Define Exact differential equation of first order and first degree.
State and prove the necessary and sufficient condition for $M(x,y)dx + N(x,y)dy = 0$ to be exact.

(2) Prove that the integrating factor of the first order linear equation $\frac{dy}{dx} + p(x)y = q(x)$ is $e^{\int p(x)\ dx}$ and its solution is $y\ e^{\int p(x)dx} = \int e^{\int p(x)dx} q(x)\ dx + c$.

(3) Show that the substitution $v = y^{1-n}$, reduces the Bernoulli's differential equation $\frac{dy}{dx} + Py = Qy^n$,
(where $n \neq 0, 1$ and P, Q are continuous functions of x on an interval I) to a linear first order O.D.E. in the variables x and v. Hence solve the O.D.E. $\frac{dy}{dx} + \frac{2}{x}\ y = 3x^2\ y^{\frac{4}{3}}$.

(4) Define the following terms:
(a) Exact Differential Equation of first order first degree. (b) Integrating Factor.
Also state 4 rules to find an integrating factor of a non-exact O.D.E.

xxxxxxxxxxxx

Chapter 2

(USMT 102) ALGEBRA I

2.1 Practical 2.1: Division Algorithm and Euclidean Algorithm

2.1.1 Prerequisite of Practical 2.1:

(1) **Division Algorithm** : Given integers a and b, with $b > 0$, there exist unique integers q and r satisfying $a = bq + r, \quad 0 \leq r < b$. The integers q and r are called, respectively, the quotient and remainder in the division of a by b.

(2) If a and b are integers, with $b \neq 0$, then there exist unique integers q and r such that $a = qb + r, \ 0 \leq r < |b|$.

(3) An integer b is said to be **divisible** by an integer $a \neq 0$, in symbol $a \mid b$, if there exists some integer c such that $b = ac$.
We write $a \nmid b$ to indicate that b is not divisible by a.

(4) For integers a, b, c, the following hold: (divisors are assumed to be nonzero)

(i) $a \mid 0, 1 \mid a, a \mid a$.
(ii) $a \mid 1$ if and only if $a = \pm 1$.
(iii) If $a \mid b$ and $c \mid d$, then $ac \mid bd$.
(iv) $a \mid b$ and $b \mid a$ if and only if $a = \pm b$.
(v) If $a \mid b$ and $b \neq 0$, then $|a| \leq |b|$.
(vi) If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$ for arbitrary $x, y \in \mathbb{Z}$.

(5) **Greatest Common Divisor**: Let a and b be given integers, with atleast one of them different from zero. The greatest common divisor of a and b, denoted by gcd (a, b), is the positive integer d satisfying
(1) $d \mid a$ and $d \mid b$.
(2) if $c \mid a$ and $c \mid b$, then $c \leq d$.

(6) Given integers a and b, not both of which are zero, there exist integers x and y such that gcd $(a, b) = ax + by$.

(7) If a and b are given integers, not both zero, then the set $T = \{ax + by : x, y \in \mathbb{Z}\}$ is precisely the set of all multiples of $d =$ gcd (a, b).

(8) Two integers a and b not both of which are zero, are said to be **relatively prime** whenever $\gcd(a, b) = 1$.

(9) Let a and b be integers, not both zero. Then a and b are relatively prime if an only if there exist integers x and y such that $1 = ax + by$.

(10) If gcd $(a, b) = d$, then $\gcd(a/d, b/d) = 1$.

(11) If $a \mid c$ and $b \mid c$ with $\gcd(a, b) = 1$, then $ab \mid c$.

(12) **Euclid's Lemma**: If $a \mid bc$, with gcd $(a, b) = 1$, then $a \mid c$.

(13) Let a, b be integers, not both zero. For a positive integer $d, d =$ gcd (a, b) if and only if
(1) $d \mid a$ and $d \mid b$.
(2) whenever $c \mid a$ and $c \mid b$, then $c \mid d$.

(14) If $k \in \mathbb{Z}, k > 0$, then $\gcd(ka, kb) = k \gcd(a, b)$.

(15) For any integer $k \neq 0, \gcd(ka, kb) = |k| \gcd(a, b)$.

(16) The **Least common multiple** of two nonzero integers a and b, denoted by lcm (a, b), is the positive integer m satisfying
(1) $a \mid m$ and $b \mid m$,
(2) If $a \mid c$ and $b \mid c$, with $c > 0$, then $m \leq c$.

(17) For positive integers a and $b, \gcd(a, b) * \text{lcm}(a, b) = ab$.

2.1.2 PRACTICAL 2.1

(A) Objective Questions

(1) If $n = 7^3 * 5^4 * 3^5, m = 105 * 10^5$ then gcd $(n, m) =$

(a) 2625 (b) 2645 (c) 1 (d) None of these

(2) When a number is divided by 893 the reminder is 193. What will be the reminder when it is divided by 47?

(a) 19 (b) 5 (c) 33 (d) 23

(3) $a + b = 156, (a, b) = 13$. The number of such pairs is..........

(a) 2 (b) 5 (c) 4 (d) 3

(4) If $(a, b) = 2, (b, 4) = 2$ then $(a + b, 4) =$ ....

(a) 1 (b) 2 (c) 4 (d) None of these

(5) If gcd $(a, b) =$ lcm (a, b) then the following is true.

(a) $a > b$ (b) $a = b$ (c) $a + b = 1$ (d) None of these

(6) Which of the following is true

(i) $\frac{a(a+1)}{2}$ (iii) $\frac{a(a+1)(a+3)}{4}$

(ii) $\frac{a(a+1)(a+2)}{3}$ (iv) $\frac{a(a+1)(a+2)}{4}$

(a) (i) (c) (i) and (ii) both

(b) (ii) (d) (iii) and (iv) both

(7) gcd of 12006 and -975 is

(a) 1 (b) -1 (c) 4 (d) None of these

(8) For any $n \in \mathbb{N}$. $(55n + 2, 22n + 1) =$ ....

(a) 11 (b) 0 (c) 1 (d) None of these

(9) $a_1 = 4, a_n = 4a_{n-1}, n > 1$ then a_{100} mod $7 =$ ....

(a) 2 (b) 3 (c) 4 (d) 5

(10) If the least common multiple of two nonzero integers a and b, is m then

(a) $m|a$ and $m|b$ (b) $a|m$ and $b|m$ (c) $m = ab$ (d) None of these

(11) If the least common multiple of two nonzero integers a and b, is m. If $a|c$ and $b|c$, with $c > 0$, then

(a) $m = c$ (b) $m > c$ (c) $m \leq c$ (d) None of these

(12) Given integers a and b, with $b > 0$ then

(a) $a = qb + r, 0 \leq r \leq b$ (c) $a = qb + r, 0 \leq r < a$

(b) $a = qb + r, 0 \leq r < 1$ (d) None of these

(13) If If $a \mid 1$, then

(a) $a = \pm 1$ (b) $a = 0$ (c) $a = 1$ (d) None of these

(14) If $a \mid b$ and $b \mid a$ then

(a) $a = b$ (b) $a = \pm b$ (c) $a = b = 0$ (d) None of these

(15) $a \mid b$ and $a \mid c$ then

(a) $a \mid bx + cy$ (b) $a \mid 1$ (c) $a = 1$ (d) None of these

(16) If $a \mid b$ with $b \neq 0$

(a) $|a| = |b|$ (b) $|a| > |b|$ (c) $|a| \leq |b|$ (d) None of these

(17) Let $\gcd(a, b) = d$, then

(a) $d \mid a$ only (b) $d \mid b$ only (c) $d \mid a$ and $d \mid b$ (d) None of these

(18) Let $\gcd(a, b) = d$. If $c \mid a$ and $c \mid b$, then

(a) $c \leq d$ (b) $c \geq d$ (c) $c = 1$ (d) None of these

(B) Descriptive Questions

(1) For integers a, b and c prove the following

(i) $(a, (b, c)) = ((a, b), c)$
(ii) $9|b + c$ and $9|b - c \Rightarrow 9|50b - 22c$.
(iii) $(a, c) = d, a|b, c|b \Rightarrow ac|bd$
(iv) $a|c, b|c$ and $(a, b) = 1 \Rightarrow ab|c$
(v) $(a, b) = 1$ and $a|bc \Rightarrow a|c$
(vi) $2|a(a + 1)$ and $3|a(a + 1)(a + 2)$
(vii) $(a, 4) = 2, (b, 4) = 2 \Rightarrow (a + b, 4) = 4$.

(2) Show that any integer of the form $6k + 5$ is also fo the form $3k + 2$, but not conversely.

(3) Use the Division Algorithm to establish that

(i) Every odd integer is either of the form $4k + 1$ or $4k + 3$.
(ii) the square of any integer is either of the form $3k$ or $3k + 1$.
(iii) the cube of any integer ie either of the form $9k, 9k + 1$ or $9k + 8$.

(4) Prove that no integer in the sequence $11, 111, 1111, 11111, \ldots$ is a perfect square. (Hint $11 = 8+3, 111 = 108+3, 1111 = 1000+108+3, 11111 = 10^4+10^3+108+3, 111111 = 10^5 + 10^4 + 10^3 + 108 + 3 \cdots$. Each is of the form $4k + 3$. But square of every integer is either of the form $4k$ or $4k + 1$.)

(5) Prove or disprove: if $a \mid (b + c)$, then either $a \mid b$ or $a \mid c$.

(6) Prove that, for any integer a, one of the integers $a, a + 2, a + 4$ is divisible by 3. [Hint: By the Division Algorithm the integer a must be of the form $3k$ or $3k + 1$ or $3k + 2$.]

(7) Prove that the sum of the squares of two odd integers cannot be a perfect square.

(8) Show that the difference between the cubes of two consecutive integers is never divisible by 2.

(9) Prove that, for a positive integer n and any integer a, $\gcd(a, a+n)$ divides n; hence $\gcd(a, a+1) = 1$.

(10) Given integers a and b, prove that there exist integers x and y for which $c = ax + by$ if and only if $\gcd(a, b)|c$.

(11) Prove that if gcd $(a, b) = 1$ and gcd $(a, c) = 1$, then gcd $(a, bc) = 1$.

(12) If gcd $(a, b) = 1$ and $c \mid a$ then prove that gcd $(c, b) = 1$.

(13) If gcd $(a, b) = 1$ then prove that gcd $(ac, b) =$ gcd (c, b).

(14) If gcd $(a, b) = 1$ and $c \mid (a + b)$ then gcd $(a, c) =$ gcd $(b, c) = 1$.

(15) Find gcd $(143, 227)$, gcd $(306, 657)$ and gcd $(272, 1479)$.

(16) Use the Euclidean Algorithm to obtain integers x and y satisfying

(i) gcd $(56, 72) = 56x + 72y$.

(ii) gcd $(24, 138) = 24x + 138y$.

(iii) gcd $(119, 272) = 119x + 272y$.

(iv) gcd $(1769, 2378) = 1769x + 2378y$.

(17) Find lcm $(143, 227)$, lcm $(306, 657)$ and lcm $(272, 1479)$.

(18) Prove the following

(i) GCD of 2 consecutive integers is 1

(ii) GCD of 2 consecutive odd integers is 1

(iii) GCD of 2 consecutive even integers is 2

(iv) $(ka, kb) = |k|(a, b)$ for any integer k

(19) For any natural number n prove that the following pairs are relatively prime.

(i) $2n + 1, 9n + 4$

(ii) $5n + 2, 7n + 3$

(iii) $55n + 2, 22n + 1$

(iv) $21n + 4, 14n + 3$

(20) The Fibonacci no. F_n is defined as $F_1 = 1, F_2 = 2, F_{n+1} = F_n + F_{n-1}\ \forall n \geq 2$. Then prove that $(F_n, F_{n+1}) = 1$.

(21) Prove that if for integers $a, b, (a, b) = 1$ then

(i) $(a + b, a - b) = 1$ or 2

(ii) $(a + b, a^2 + b^2) = 1$ or 2

(iii) $(a^2, b^2) = 1$

(iv) $(2a + b, a + 2b) = 1$ or 3

(22) Find g c d, expresses it in form $ma + nb$.

2.1

(i) 143, 227
(ii) −1975, −851
(iii) 725, 441
(iv) 325, 26
(v) −1122, 215
(vi) 567, 123
(vii) 3467, 1123
(viii) 560, 124
(ix) 45, −9012
(x) 879, 1216
(xi) 12006, −1975
(xii) 432, 127

(23) For $a, b \neq 0$ prove $(a, b) = [a, b]$ iff $a = b$.

(24) Find positive integers a and b such that $(a, b) = 10$ and $[a, b] = 100$.

(25) Find LCM of 482 and 1687.

xxxxxxxx

2.2 Practical 2.2: Prime numbers, Fundamental Theorem of Arithmetic, Congruence

2.2.1 Prerequisite of Practical 2.2

(1) An integer $p > 1$ is called a **prime number**, or a prime, if its only positive divisors are 1 and p.
An integer greater than 1 which is not a prime is called **composite**.

(2) If p is a prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

(3) If p is a prime and $p \mid a_1 a_2 \cdots a_n$, then $p \mid a_k$ for some $k, 1 \leq k \leq n$.

(4) If $p, q_1, q_2, \cdots, q_n$ are all primes and $p \mid q_1 q_2 \cdots q_n$, then $p = q_k$ for some k, where $1 \leq k \leq n$.

(5) **Fundamental Theorem of Arithmetic**: Every positive integer $n > 1$ can be expressed as a product of primes; this representation is unique, apart from the order in which the factors occur.

(6) Any positive integer $n > 1$ can be written uniquely in a canonical form $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$, where, for $i = 1, 2, \cdots, r$, each k_i is a positive integer and each p_i is a prime with $p_1 < p_2 < \cdots < p_r$.

(7) **Euclid's Theorem**: There are an infinite number of primes.

(8) There is an infinite number of primes of the form $4n + 3$.

(9) There is an infinite number of primes of the form $4n + 1$.

(10) There is an infinite number of primes of the form $6n - 1$.

(11) Let n be a positive integer. Two integers a and b are said to be **congruent modulo** n, written as $a \equiv b (\text{mod } n)$ if n divides the difference $a - b$; that is, $a - b = kn$ for some integer k.

(12) For arbitrary integers a and $b, a \equiv b(\text{mod } n)$ if and only if a and b leave the same nonnegative remainder when divided by n.

(13) Let $n > 0$ be a fixed integer and a, b, c, d be arbitrary integers. Then the following properties hold:

(i) $a \equiv a(\text{mod } n)$.

(ii) If $a \equiv b(\text{mod } n)$, then $b \equiv a(\text{mod } n)$.

(iii) If $a \equiv b(\text{mod } n)$ and $b \equiv c(\text{mod } n)$ then $a \equiv c(\text{mod } n)$.

(iv) If $a \equiv b(\text{mod } n)$ and $c \equiv d(\text{mod } n)$ then $a + c \equiv b + d(\text{mod } n)$ and $ac \equiv bd(\text{mod } n)$.

(v) If $a \equiv b(\text{mod } n)$, then $a + c \equiv b + c(\text{mod } n)$ and $ac \equiv bc(\text{mod } n)$.

(vi) If $a \equiv b(\text{mod } n)$, then $a^k \equiv b^k(\text{mod } n)$ for any positive integer k.

(14) If $ca \equiv cb(\text{mod } n)$, then $a \equiv b(\text{mod } n/d)$ where $d = \gcd(c, n)$.

(15) If $ca \equiv cb(\text{mod } n)$ and $\gcd(c, n) = 1$, then $a \equiv b(\text{mod } n)$.

(16) If $ca \equiv cb(\text{mod } p)$ and $p \not| c$, where p is a prime number, then $a \equiv b(\text{mod } p)$.

(17) An equation of the form $ax \equiv b(\mod n)$ is called a **linear congruence**, and an integer x_0 is called a solution of $ax \equiv b(\mod n)$ if $ax_0 \equiv b(\mod n)$.

(18) The linear congruence $ax \equiv b(\mod n)$ has a solution if and only if $d \mid b$, where $d =$ gcd (a, n). If $d \mid b$, then it has d mutually incongruent solutions modulo n.

(19) Consider the linear congruence $ax \equiv b(\mod n)$ such that $d \mid b$, where $d =$ gcd (a, n). If x_0 is one solution of the congruence $ax \equiv b(\mod n)$, then all the d incongruent solutions are $x_0, x_0 + \frac{n}{d}, x_0 + 2 * \frac{n}{d}, \cdots, x_0 + (d-1) * \frac{n}{d}$.

(20) If gcd $(a, n) = 1$, then the linear congruence $ax \equiv b(\mod n)$ has a unique solution modulo n.

(21) Suppose we want to solve the linear congruence $6x \equiv 15 \mod 21$.

Step I Find gcd(a, n). Here $\gcd(a, n) = \gcd(6, 21) = 3$.

Step II Check whether $d \mid b$ or not. Here $d = 3, b = 15$ and hence $d \mid b$.

Step III If no, then the linear congruence doesn't have a solution. We stop here.

Step IV If yes, then, we know that there are d incongruent solutions modulo n. Here, $d = 3 \implies 6x \equiv 15 \mod 21$ has 3 incongruent solutions modulo 15.

Step V Divide the given linear congruence by d. So, we divide by 3. We will get the equivalent congruence $2x \equiv 5 \mod 7$. Since $(2, 7) = 1$, this congruence has unique solution modulo 7. Find that solution. $x \equiv 6 \mod 7$ is the unique solution of this congruence. We select $x_0 = 6$.

Step VI Original congruence was modulo 21. Its all $d = 3$ incongruent solutions modulo 21 are obtained by taking $t = 0, 1, 2$, in the formula $x = x_0 + t * \frac{n}{d}$. The 3 incongruent solutions modulo 21 are $6, 6 + 1 * 7, 6 + 2 * 7$, that is, $6, 13, 20$.

2.2

Step VII In terms of congruence, the 3 incongruent modulo 21 solutions are $x \equiv 6 \mod 21, x \equiv 13 \mod 21, x \equiv 20 \mod 21$.

Step VIII Rough Work: We substitute each of $6, 13, 20$ in the given congruence and verify.

Put $x = 6$ in L.H.S. of $6x \equiv 15 \mod 21$.

$$\begin{aligned} 6x &\equiv 6 * 6 \mod 21 \\ &\equiv 36 \mod 21 \\ &\equiv 15 \mod 21 \end{aligned}$$

Put $x = 13$ in L.H.S. of $6x \equiv 15 \mod 21$.

$$\begin{aligned} 6x &\equiv 6 * 13 \mod 21 \\ &\equiv 6 * (-8) \mod 21 \\ &\equiv -48 \mod 21 \\ &\equiv -6 \mod 21 \\ &\equiv 15 \mod 21 \end{aligned}$$

Put $x = 20$ in L.H.S. of $6x \equiv 15 \mod 21$.

$$\begin{aligned} 6x &\equiv 6 * 20 \mod 21 \\ &\equiv 6 * (-1) \mod 21 \\ &\equiv -6 \mod 21 \\ &\equiv 15 \mod 21 \end{aligned}$$

Note: $x \equiv 6 \mod 21$ implies $x \in \{\cdots, -26, -15, 6, 27, 31, \cdots\}$. Similarly $x \equiv 13 \mod 21$ includes $x \in \{\cdots, -29, -8, 13, 34, 55, \cdots\}$ and so on.

2.2.2 PRACTICAL 2.2

(A) Objective Questions

Choose correct alternative in each of the following:

(1) Which one of the following is a prime number?

(a) 567 (b) 451 (c) 701 (d) None of these

(2) If p is a prime such that $p \mid (11 * 17 * 23)$ and $p \nmid 391$ then

(a) $p = 4301$ (b) $p = 17$ (c) $p = 23$ (d) $p \mid 253$

(3) If p is a prime and $p \mid (11 * 17 * 23)$ and $p \nmid 391$ then

(a) $p = 4301$ (b) $p = 17$ (c) $p = 23$ (d) $p = 11$

(4) If $p_1, p_2, p_3, q_1, q_2, q_3$ are primes and $n = p_1 * p_2 * p_3 = q_1 * q_2 * q_3$ where $p_1 \leq p_2 \leq p_3, q_1 \geq q_2 \geq q_3$ then

(a) $p_1 = q_1, p_2 = q_2, p_3 = q_3$
(b) $p_1 = q_3, p_2 = q_2, p_3 = q_1$
(c) $p_1 \neq q_1, p_2 \neq q_2, p_3 \neq q_3$
(d) None of these.

(5) If p is a prime then

(a) $\sqrt{p} = \frac{a}{b}$ for some $a, b \in \mathbb{Z}, b \neq 0$

(b) $\sqrt{p} \in \mathbb{Z}$.

(c) $\sqrt{p}$ is irrational

(d) None of these.

(6) If $n^3 - 1$ is a prime then $n =$

(a) 7 only

(b) 2 only

(c) 5 only

(d) there can be more than one solution.

(7) If p is a prime of the form $3n+1, n \geq 1$ then p is also of the form

(a) $4m+3, m \geq 1$ (b) $6m+1, m \geq 1$ (c) $5m+2, m \geq 1$ (d) None of these

(8) If n is of the form $3m+2$ then n has a prime factor of the form

(a) $3m+1$ (b) $3m$ (c) $3m+2$ (d) None of these

(9) If gcd $(a, b) = p$ where p is a prime, then $\gcd(a^2, b)$ is

(a) p or p^2 (b) p (c) p^2 (d) None of these

(10) $25 + 37 \equiv\ ?\ \mod 12$.

(a) 11 (b) 0 (c) 62 (d) None of these

(11) $23 * 43 \equiv\ ?\ \mod 8$

(a) 5 (b) 12 (c) 6 (d) None of these

(12) If $9x \equiv 1 \mod 13$ then $x =$

(a) $x \equiv 3 \mod 13$ (b) $x \equiv 4 \mod 13$ (c) $x \equiv 5 \mod 13$ (d) None of these

(13) $2^{20} \equiv\ ?\ \mod 41$

(a) 1 (b) 2 (c) 3 (d) None of these

(14) $1! + 2! + 3! + \cdots + 100! \equiv\ ?\ \mod 12$

(a) 7 (b) 8 (c) 9 (d) None of these

(15) If $a \equiv b \mod n$ and $m \mid n$ then $a \equiv\ ?\ \mod m$

(a) bn (b) Y (c) bm (d) None of these

(16) If $a \equiv b \mod n$ and $c > 0$ then $ca \equiv\ ?\ \mod n$

(a) $\frac{c}{b}$ (b) $\frac{c}{a}$ (c) ca (d) None of these

(17) The number of mutually incongruent solutions modulo 30 of $9x \equiv 31 \mod 30$ is

(a) 3 (b) 1 (c) 5 (d) None of these

(B) Descriptive Questions

(1) Solve the following linear congruences:

(i) $25x \equiv 15(\mod 29)$.
(ii) $9x \equiv 21(\mod 30)$.
(iii) $5x \equiv 2(\mod 26)$.
(iv) $36x \equiv 8(\mod 102)$.
(v) $34x \equiv 60(\mod 98)$.
(vi) $140x \equiv 133(\mod 301)$

(2) Find all prime p and q such that

(i) $p - q = 3$.
(ii) $p^2 - 2q^2 = 1$.

(3) If p and $p^2 + 2$ are prime show that $p^3 + 2$ is prime.

(4) If $2^n + 1$ is an odd prime for some integers n then prove that n is power of 2.

(5) If p is a prime of the form $3n+1, n \geq 1$ then show that p is also of the form $6m+1, m \geq 1$.

(Hint : $p = 3n+1$ for some $n \in \mathbb{Z}$. Now $n = 2k$ or $n = 2k+1$. If $n = 2k+1$ then $p = 6k+4$ but then $2|p$ and so $p = 2$ contradiction as $p = 3n + 1$.)

(6) Prove that each integer of the form $3n + 2$ then n has a prime factor of this form.
(Hint: Let $a = 3m + 1$ for some $m \in \mathbb{Z}$. Let $a = p_1 p_2 \cdots p_r$ be the prime factorization of a. If $p_i = 3$ then $3 \mid a \implies 3 \mid (3n+2)$. This is a contradiction. Every p_i is of the form $3n+1$ then the product of these factor will also be of the form $3n + 1$. This is a contradiction. Hence there exists at least one prime factor of a of the form $3n + 2$.)

(7) Prove that the only prime of the form $n^3 - 1$ is 7.
(Hint: $p = n^-1 \implies p = (n-1)(n^2 + n + 1)$. But p is a prime. Hence every divisor of p must be either 1 or p. If $n - 1 = p$ then $n = p + 1$ and hence $p = (p + 1)^2 - 1$. We get a contradiction here. Hence $n - 1 - 1$.)

(8) Prove that the only prime p for which $3p + 1$ is a perfect square is $p = 5$.
(Hint: $3p + 1 = x^2$. So, $x \neq 0$. Also, $3p = x^2 - 1 = (x - 1)(x + 1)$. So $3 \mid ((x - 1)(x + 1))$. As 3 is a prime, $3 \mid (x - 1)$ or $3 \mid (x + 1)$. One case gives a contradiction.)

(9) Prove that $n^4 + 4$ is composite for each $n > 1$.

(10) Find all primes that divide 50!.

(11) Let n be a positive integer and $p_1, p_2, \cdots, p_n$ be prime numbers greater than 5 such that $6 \mid p_1^2 + p_2^2 + \cdots + p_n^2$ show that $6 \mid n$.

(12) Prove that there are no integers x, y satisfying $(x, y) = 3$ and $x + y = 100$.

(13) Prove that $4 \nmid a^2 + 2$ for any integer a.

(14) Prove or disprove

(i) If p is a prime, $p \mid a$ and $p \mid a^2 + b^2$ then $p|b$.

(ii) If $a^2 \mid c^3$ then $a \mid c$

(15) If p and q are primes ≥ 5 prove $24 \mid p^2 - q^2$.

(16) Show that $\sqrt[3]{100}$ is irrational.

(17) Prove the following

(i) $10! \equiv -1(\text{mod } 11)$

(ii) $18! \equiv -1 \text{ mod}(437)$

(iii) $7 \mid 2222^{5555} + 5555^{2222}$

(iv) $88^{641} \equiv 3(\text{mod } 86)$

(v) $89 \mid 2^{44} - 1$

(vi) $2^340 \equiv 1(\text{mod } 31)$

(vii) $3334^{24} \equiv 1(\text{mod } 7)$.

xxxxxxxxxxx

2.3 Practical 2.3: Functions, Bijective and Invertible functions, Composition of functions

2.3.1 Prerequisite for Practical 2.3

(1) **Cartesian Product**: If X and Y are two non-empty sets then the **cartesian product** of X and Y is denoted by $X \times Y$ and is defined as $X \times Y = \{(x, y) : x \in X \text{ and } y \in Y\}$. The pair (x, y) is called as an ordered pair.

(2) If X and Y are finite sets having m and n elements respectively, then $X \times Y$ is a set having $m \times n$ elements.

(3) If X and Y are two non-empty sets then a **relation** from X to Y is a subset of $X \times Y$.
If we denote a relation by R then $R \subseteq X \times Y$.
For example: If $X = \{a, b\}$ and $Y = \{1, 2, 3\}$, then $X \times Y = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$.
Let $R_1 = \{(a, 1), (b, 3), (a, 3)\}$. Then R_1 is a relation from X to Y. We say that a is related to 1 and 3, and b is related to 3 under R_1.
Let $R_2 = \{(a, 2), (b, 3)$. Then R_2 is a relation from X to Y. We say that a is related to 2 and b is related to 3 under R_2.

(4) **Function**: Let X and Y be two non-empty sets. A **function** from X to Y is a subset f of $X \times Y$ with the property that for each $x \in X$, there is a unique $y \in Y$ such $(x, y) \in f$. The set X is called the **domain** of f and Y the **codomain** of f.
Usually, we write $f : X \longrightarrow Y$ to indicate that f is a function from X to Y. Also, instead of $(x, y) \in f$, we write $y = f(x)$, and call $f(x)$ the value of f at x. This is also indicted by writing $x \mapsto f(x)$, and saying that f maps x to $f(x)$.

(5) Both, a relation and a function, are subsets of $X \times Y$ and every function $f : X \longrightarrow Y$ is a relation. But a relation need not be a function. Above relation R_1 is not a function as a is related to two elements from Y under R_1.

(6) Let $f : X \longrightarrow Y$ is a function. The **range** of f is defined as $f(X) = \{f(x) : x \in X\}$. Also, if $y = f(x)$ then y is called the **image** of x and x is called a **pre-image** of y.

(i) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^2$. Here domain of $f = \mathbb{R}$, codomain of $f = \mathbb{R}$ and range of $f = \{f(x) : x \in \mathbb{R}\} = \{x^2 : x \in \mathbb{R}\} = [0, \infty)$.
For $x = 2 \in \mathbb{R}$ (domain of f) $f(2) = 4$. So 4 is the image of 2. 2 is a pre-image of 4. Since $f(-2)$ is also 4, we have, $2, -2$ are two pre-images of 4.
Now consider $y = -3 \in \mathbb{R}$(codomain of f). There is no $x \in \mathbb{R}$ (domain of f) such that $x^2 = -3$. Hence -3 does not have a pre-image.

(ii) Let X, Y be not empty sets and $c \in Y$. Define a function $f : X \longrightarrow Y$ as $f(x) = c$ for all $x \in X$. Then f is called a constant function.

(iii) **Absolute value function**: $|\ | : \mathbb{R} \longrightarrow \mathbb{R}, |x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$
We can verify that 4 and -4 are two pre-images of 4 and -5 does not have any pre-image in the domain $\mathbb{R}$.

(iv) **Identity function**: Let X be a non-empty set. Let i_X be a function defined as $i_X : X \longrightarrow X, i_X(x) = x$. Then i_X is called the **identity function** on X. Here every element in X (codomain of i_X) has exactly one pre-image in X (domain of i_X).
Domain of $i_X = X$, codomain of $i_X = X$ and range of $i_X = X$.

(v) **Projection function**: Let X and Y be non-empty sets. The projection function π_X and π_Y are defined as
$\pi_X : X \times Y \longrightarrow X, \pi_X\Big((x, y)\Big) = x$ and
$\pi_Y : X \times Y \longrightarrow Y, \pi_Y\Big((x, y)\Big) = y$.
Range of $\pi_X = X$ and Range of $\pi_Y = Y$.

(vi) **Floor function**: The floor function $\lfloor x \rfloor : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as $\lfloor x \rfloor =$ greatest integer $\leq x$. For example $\lfloor 1.5 \rfloor = 1, \lfloor -2.7 \rfloor = -3$.

(vii) **Ceiling function**: The ceiling function $\lceil x \rceil : \mathbb{R} \longrightarrow \mathbb{R}$ s defined as $\lceil x \rceil =$ least integer $\geq x$. For example $\lceil 1.5 \rceil = 2, \lceil -2.7 \rceil = -2$.

(viii) **Characteristic function**: Let X be a non empty set and $A \subseteq X$. Define $f : X \longrightarrow \mathbb{R}$ as follows: $f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$ Then f is called as a characteristic function on X.

(7) **Injective or one-one function** : A function $f : X \longrightarrow Y$ is said to be **injective or one-one** if f maps distinct points to distinct points, that is, $x_1, x_2 \in X, f(x_1) = f(x_2) \Longrightarrow x_1 = x_2$.

For example, the identity i_X is injective but the absolute value function is not injective. Projection maps are not injective.

(8) **Surjective or onto function**: A function $f : X \longrightarrow Y$ is said to be **surjective or onto** if every element in Y has at least one pre-image in X. That is, f is surjective if $f(X) = Y$. For example, the identity i_X, projections maps are surjective but the absolute value function, floor function, ceiling function as defined above are not surjective. Also above f is not surjective as $7 \in Y$ does not have a pre-image in X.

(9) **Bijective function**: A function $f : X \longrightarrow Y$ is said to be **bijective** if it is injective and surjective.
The identity function defined as $i_X : X \longrightarrow X, i_X(x) = x$ for all $x \in X$, is bijective.

(10) **Equality of two functions**: Let $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ be two functions. We say that the function f is equal to the function g if $f(x) = g(x)$ for all $x \in X$, and write as $f = g$ on X.

(11) **Composite function**: If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are functions. Then the function $h : X \longrightarrow Z$, defined by $h(x) = g\Big(f(x)\Big)$ for all $x \in X$, is called the **composite** of f and g and is denoted by $g \circ f$.

Remarks 2.3.1

(i) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ are functions such that $g \circ f = i_X$ then f is injective.
Suppose $f(x_1) = f(x_2)$ for some $x_1, x_2 \in X$. Then $g(f(x_1) = g(f(x_2))$, that is, $(g \circ f)(x_1) = (g \circ f)(x_2)$. But $g \circ f = i_X$. So, $(g \circ f)(x_1) = x_1$ and $(g \circ f)(x_2) = x_2$. Hence $x_1 = x_2$. So, f is injective.

(ii) If $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ are functions such that $f \circ g = i_Y$ then f is surjective.
Let $y \in Y$. As $g : Y \longrightarrow X, g(y) \in X$. Hence $g(y) = x$ for some $x \in X$. We will show that $f(x) = y$.
Since $g(y) = x$, we apply f on both sides. So, $f(g(y)) = f(x)$. That is, $(f \circ g)(y) = f(x)$. But $f \circ g = i_Y$. Hence $(f \circ g)(y) = y$. Hence $y = f(x)$. Thus y has a pre-image under f. Hence f is surjective.

(iii) If $f : X \longrightarrow Y, g : Y \longrightarrow X$ are two functions such that $g \circ f = i_X$ then f is injective and g is surjective.

(iv) $f : \mathbb{N} \longrightarrow \mathbb{Z} \setminus \{0\}$ defined as $f(x) = x$. Let $g : \mathbb{Z} \setminus \{0\} \longrightarrow \mathbb{N}$ defined as $g(x) = |x|$.
We will show that $g \circ f = i_{\mathbb{N}}$.
$g \circ f(x) = g(f(x)) = g(x) = |x| = x$ as $x \in \mathbb{N}$, so $|x| = x$.
Hence by above remark, f is one-one.
Observe that this f is not surjective.

(v) $f : \mathbb{R} \longrightarrow [0, \infty)$ defined as $f(x) = x^2$. Let $g : [0, \infty) \longrightarrow \mathbb{R}$ defined as $g(x) = \sqrt{x}$.
We will show that $f \circ g = i_{[0,\infty)}$.
$f \circ g(x) = f(g(x)) = f(\sqrt{x}) = (\sqrt{x})^2 = x$.
Hence by above remark, f is surjective.
Observe that this f is not injective.

(vi) We will show that $g \circ f = i_X$ need not imply that $f \circ g = i_Y$.
Consider $f : \mathbb{N} \longrightarrow \mathbb{Z} \setminus \{0\}$ defined as $f(x) = x$. Let $g : \mathbb{Z} \setminus \{0\} \longrightarrow \mathbb{N}$ defined as $g(x) = |x|$. We have shown that $g \circ f = i_{\mathbb{N}}$. We will show that $f \circ g \neq i_{\mathbb{Z}\setminus\{0\}}$.

$(f \circ g)(-2) = f(g(-2)) = f(|-2|) = f(2) = 2$. So $(f \circ g)(-2) \neq -2$. Hence $f \circ g \neq i_{\mathbb{Z}\setminus\{0\}}$ This example shows that $g \circ f = i_X$ need not imply that $f \circ g = i_Y$.

(vii) We will show that his example shows that $f \circ g = i_Y$ need not imply that $g \circ f = i_X$. $f : \mathbb{R} \longrightarrow [0, \infty)$ defined as $f(x) = x^2$. Let $g : [0, \infty) \longrightarrow \mathbb{R}$ defined as $g(x) = \sqrt{x}$.
We have shown that $f \circ g = i_{[0,\infty)}$.
Now, consider $(g \circ f)(-2) = g(f(-2)) = g(4) = \sqrt{4} = 2$ (as $\sqrt{x} \geq 0$ for all $x \in [0, \infty)$)
So, $(g \circ f)(-2) \neq -2$. Hence $g \circ f = i_{\mathbb{R}}$.

(12) **Inverse function**: Let $f : X \longrightarrow Y$ be a function. A function $g : Y \longrightarrow X$ is called an **inverse function** of f if $f \circ g = i_Y$ and $g \circ f = i_X$. If such function g exists then it is unique and it is denoted by f^{-1}.

(13) Let $f : X \longrightarrow Y$ be a function. Then f is bijective if and only if f^{-1} exists.
If f is bijective, then for $y \in Y$, there is unique $x \in X$ such that $f(x) = y$. Define $g : Y \longrightarrow X$ as $g(y) = x$ where $f(x) = y$. We will verify that $f \circ g = i_Y$ and $g \circ f = i_X$.

$$(g \circ f)(x) = g(f(x)) = g(y) = x.$$
Thus $g \circ f = i_X$.

$$(f \circ g)(y) = f(g(y)) = f(x) = y.$$
Thus $f \circ g = i_Y$.

Hence the function g defined as above is the inverse function of f, that is, $g = f^{-1}$.
Conversely, suppose there exists $g : Y \longrightarrow X$ such that $g \circ f = i_X$ and $f \circ g = i_Y$.
We will show that f is bijective.

TST f is injective.
From Remark 2.3.1 no. (1),
$g \circ f = i_X \Longrightarrow f$ is injective.

TST f is surjective.
From Remark 2.3.1 no. (2),
$f \circ g = i_Y \Longrightarrow f$ is surjective.

Hence f is bijective. For example: Show that $f : \mathbb{R} \setminus \{\frac{5}{7}\} \longrightarrow \mathbb{R} \setminus \{0\}, f(x) = \dfrac{1}{7x - 5}$ is bijective and hence find f^{-1}.

To show that f is injective.
Suppose $f(x_1) = f(x_2)$
$\dfrac{1}{7x_1 - 5} = \dfrac{1}{7x_2 - 5}$.
$7x_1 - 5 = 7x_2 - 5$
$x_1 = x_2$.
Hence f is injective.

To show that f is surjective.
Let $y \in \mathbb{R} \setminus \{0\}$.
If $x \in \mathbb{R} \setminus \{\frac{5}{7}\}$ such that $f(x) = y$,
then $\dfrac{1}{7x - 5} = y$.
$\Longrightarrow 1 = y(7x - 5)$.
That is, $1 = 7xy - 5y$.
Hence $1 + 5y = 7xy$. As, $y \neq 0$, $\dfrac{1 + 5y}{7y} = x$.
So, $\dfrac{1 + 5y}{7y}$ is a pre-image of x.
Hence f is surjective.

Thus, f is bijective. Hence f^{-1} exists. It is defined as $f^{-1}:\mathbb{R}\setminus\{0\}\longrightarrow\mathbb{R}\setminus\{\frac{5}{7}\}, f^{-1}(y)=\dfrac{1+5y}{7y}$.

Show that $f:\mathbb{R}\setminus\{-\frac{7}{5}\}\longrightarrow\mathbb{R}\setminus\{-\frac{2}{5}\}, f(x)=\dfrac{3-2x}{5x+7}$ is bijective and hence find f^{-1}.

To show that f is injective.
Suppose $f(x_1)=f(x_2)$
$\dfrac{3-2x_1}{5x_1+7}=\dfrac{3-2x_2}{5x_2+7}$.
$(3-2x_1)(5x_2+7)=(3-2x_2)(5x_1+7)$
$15x_1x_2+21-10x_1x_2-14x_1=15x_1x_2+21-10x_1x_2-14x_2$.
$-14x_1=-14x_2$.
$\implies x_1=x_2$. Hence f is injective.

To show that f is surjective.
Let $y\in\mathbb{R}\setminus\{-frac25\}$.
If $x\in\mathbb{R}\setminus\{-\frac{7}{5}\}$ such that $f(x)=y$,
then $\dfrac{3-2x}{5x+7}=y$.
$\implies 3-2x=y(5x+7)$.
That is, $3-2x=5xy+7y$.
Hence $3-7y=5xy+2x$.
$\implies 3-7y=x(5y+2)$.
As, $y\neq-\frac{2}{5}, 5y+2\neq 0$.
Hence $\dfrac{3-7y}{5y+2}=x$.
So, $\dfrac{3-7y}{5y+2}$ is a pre-image of x.
Hence f is surjective.

Thus, f is bijective. Hence f^{-1} exists. It is defined as $f^{-1}:\mathbb{R}\setminus\{-\frac{2}{5}\}\longrightarrow\mathbb{R}\setminus\{-\frac{7}{5}\}, f^{-1}(y)=\dfrac{3-7y}{5y+2}$.

(14) Let $f:X\longrightarrow Y$ be a function and let A be a non-empty subset of X. Then

$$f(A)=\{f(x):x\in A\}.$$

Note: $f(A)\subseteq Y$ for every $A\subseteq X$.

(15) Let $A_1,A_2\subseteq X$. Then following properties hold.

(i) $A_1\subseteq A_2\implies f(A_1)\subseteq f(A_2)$. Converse not true.
(ii) $f(A_1\cup A_2)=f(A_1)\cup f(A_2)$.
(iii) $f(A_1\cap A_2)\subseteq f(A_1)\cap f(A_2)$. The equality holds if and only if f is injective.

(16) Let $f:X\longrightarrow Y$ be a function and let B be a non-empty subset of Y. Then

$$f^{-1}(B)=\text{the set of all pre-images of elements of } B=\{x\in X:f(x)\in B\}.$$

Note: (1) Here f^{-1} indicates only that we are finding the pre-images of the elements of B. f^{-1} is not considered as a function. So, we can find $f^{-1}(B)$ irrespective of whether our function f is bijective or not.
(2) $f^{-1}(B)\subseteq X$ for every $B\subseteq Y$.

(17) Let $B_1,B_2\subseteq Y$. Then following properties hold.

(i) $B_1\subseteq B_2\implies f^{-1}(B_1)\subseteq f^{-1}(B_2)$. Converse not true.
(ii) $f^{-1}(B_1\cup B_2)=f^{-1}(B_1)\cup f^{-1}(B_2)$.

(iii) $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

(18) Let $f : X \longrightarrow Y$ be a function and let $A \subseteq X$ and $B \subseteq Y$. Then the following properties hold.

(i) $A \subseteq f^{-1}\Big(f(A)\Big)$. The equality holds if and only if f is injective.

(ii) $f\Big(f^{-1}(B)\Big) \subseteq B$. The equality holds if and only if f is surjective.

2.3.2 PRACTICAL 2.3

(A) Objective Questions:

Choose correct alternative in each of the following:

(1) Let A and B be two non empty sets. A function from A to B is a

(a) Relation which assigns atleast one element of A to a unique element of B.
(b) Relation which assigns every element of A to a unique element of B.
(c) Relation which assigns each element of A to a more than one element of B.
(d) None of these.

(2) $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ then which of the following relations is a function from A to B.

(a) $R = \{(1, a), (1, b), (2, c), (3, d), (3, a)\}$.
(b) $R = \{(1, a), (2, a), (3, a)\}$.
(c) $R = \{(1, a), (2, c)\}$.
(d) $R = \{(1, a), (2, b), (3, c), (3, d)\}$.

(3) Let $X = \{a, b, c, d, e\}$, $Y = \{1, 2, 3\}$. Which of the following relations is not a function from X to Y

(a) $R = \{(a, 1), (b, 2), (c, 3), (d, 1), (e, 2)\}$
(b) $R = \{(a, 1), (b, 1), (c, 1), (d, 1), (e, 1)\}$
(c) $R = \{(a, 1), (a, 2), (b, 2), (d, 3), (e, 3), (c, 1)\}$
(d) $R = \{(a, 3), (b, 3), (c, 2), (d, 1), (e, 1)\}$

(4) Which of the following is not a function?

(a) $f : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$, $f(n)$ = last digit of n for all $n \in \mathbb{N}$.
(b) $f : \mathbb{N} \longrightarrow \mathbb{N}$, $f(n)$ = sum of digits of n for all $n \in \mathbb{N}$.
(c) $f : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$, $f(n)$ = number of digits of n for all $n \in \mathbb{N}$.
(d) $f : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$, $f(n) = n - 1$ for all $n \in \mathbb{N}$.

(5) Let X and Y be two non empty sets.
(i) $f : X \longrightarrow Y$ is function if every element of X has a unique image in Y.
(ii) $f : X \longrightarrow Y$ is function if for $x_1, x_2 \in X$, $x_1 = x_2 \implies f(x_1) = f(x_2)$.

(a) Only (i) is true.
(b) Only (ii) is true.
(c) Both (i) and (ii) are true.
(d) Neither (i) nor (ii) is true.

(6) Let $f : X \longrightarrow Y$ be a function.

(i) Range f is a collection of those elements of Y that have atleast one pre-image in X.
(ii) Range f is a collection of images of all the elements of X.
(iii) Range $f = \{f(x)|x \in X\}$.

(a) Only (i) is true.
(b) Only (ii) is true.
(c) Only (iii) is true.
(d) All of (i), (ii), (iii) are true.

(7) $g : \mathbb{R} \longrightarrow \mathbb{R}$, $g(x) = e^x$ for all $x \in \mathbb{R}$ then Range g is

(a) $\mathbb{R}^+$ (b) $\mathbb{R}^-$ (c) $\mathbb{R}^+ \cup \{0\}$ (d) $\mathbb{R}$

(8) $h : \mathbb{R} \longrightarrow \mathbb{R}$, $h(x) = \sin x + 3$ for all $x \in \mathbb{R}$ then Range h is

(a) $[-1, 1]$ (b) $[2, 4]$ (c) $[-4, -2]$ (d) $[-4, 4]$

(9) $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \sin(x\pi)$ for all $x \in \mathbb{R}$ then Range f is

(a) $[-1, 1]$ (b) $[0, 1]$ (c) $[-1, 0]$ (d) None of these

(10) Let X, Y be two non empty sets then projection map is given by

(a) $f : X \longrightarrow X$, $f(x) = x$ for all $x \in X$.
(b) $f : X \longrightarrow X$, $f(x) = c$ for all $x \in X$.
(c) $f : X \times Y \longrightarrow X$, $f(x, y) = x + y$ for all $(x, y) \in X \times Y$.
(d) $f : X \times Y \longrightarrow Y$, $f(x, y) = y$ for all $(x, y) \in X \times Y$.

(11) Let X and Y be two non empty sets and $f : X \longrightarrow Y$ be a function.
(i) $f : X \longrightarrow Y$ is injective if no two elements of X have the same image in Y
(ii)$f : X \longrightarrow Y$ is injective if for $x_1, x_2 \in X$, $f(x_1) = f(x_2) \implies x_1 = x_2$.

(a) Only (i) is true
(b) Only (ii) is true
(c) Neither (i) nor (ii)is true
(d) Both (i) and (ii) are true.

(12) Let X and Y be two non-empty sets and $f : X \longrightarrow Y$ be a function. Suppose $A \subseteq X, B \subseteq Y$, then

(a) $f(A) \subseteq X, f^{-1}(B) \subseteq X$
(b) $f(A) \subseteq Y, f^{-1}(B) \subseteq Y$
(c) $f(A) \subseteq X, f^{-1}(B) \subseteq Y$
(d) $f(A) \subseteq Y, f^{-1}(B) \subseteq X$

(13) Let X and Y be two non empty sets and $f : X \longrightarrow Y$ be an onto function. Which is NOT TRUE?

(a) Every element of Y has atleast one pre-image in X
(b) $\{f(x)|x \in X\} = Y$
(c) Range f = Co-domain f
(d) Every element of Y has unique pre-image in X

(14) Let X and Y be two non empty sets.$f : X \longrightarrow Y$ be a function. Suppose $A \subseteq X, B \subseteq Y$. Consider the following statements.

(i) $f(A)$ is a collection of images of each element of A.
(ii) $f(B)$ is a collection of all pre-images of each element of B.
(iii) $f(A) = \{f(x) \mid x \in A\}$.
(iv) $f(B) = \{x \in X \mid f(x) \in B\}$.

(a) Only (i) and (ii) are true
(b) Only (iii) and (iv) are true
(c) All (i) to (iv) are true
(d) None of (i) to (iv) is true.

(15) $f : X \longrightarrow Y$ is a bijective function if and only if

(a) Every element of X has an unique image in Y
(b) No two elements of X have the same image in Y
(c) Every element of Y has atleast one preimage in X
(d) f is invertible.

(16) $f : \mathbb{R}\backslash\{u\} \longrightarrow \mathbb{R}\backslash\{\ v\}, f(x) = \frac{ax+b}{cx+d}$ for all $x \in \mathbb{R}\backslash\{x\}$ is bijective then,

(a) $u = \frac{-d}{c}$ and $v = \frac{a}{c}$.
(b) $u = \frac{-d}{c}$ and $v = \frac{a}{d}$.
(c) $u = \frac{-d}{c}$ and $v = \frac{b}{d}$.
(d) $u = \frac{-d}{c}$ and $v = \frac{b}{c}$.

(17) $f : \mathbb{R}\backslash\{\frac{-3}{5}\} \longrightarrow \mathbb{R}\backslash\{\frac{9}{5}\}, f(x) = \frac{9x+5}{5x+3}$, then inverse of f is

(a) $g : \mathbb{R}\backslash\{\frac{9}{5}\} \longrightarrow \mathbb{R}\backslash\{\frac{-3}{5}\}, g(y) = \frac{5-3y}{5y-9} \quad \forall y \in \mathbb{R}\backslash\{\frac{9}{5}\}$
(b) $g : \mathbb{R}\backslash\{\frac{9}{5}\} \longrightarrow \mathbb{R}\backslash\{\frac{-3}{5}\}, g(y) = \frac{3y-5}{5y-9} \quad \forall y \in \mathbb{R}\backslash\{\frac{9}{5}\}$
(c) $g : \mathbb{R}\backslash\{\frac{9}{5}\} \longrightarrow \mathbb{R}\backslash\{\frac{-3}{5}\}, g(y) = \frac{5+3y}{5y+9} \quad \forall y \in \mathbb{R}\backslash\{\frac{9}{5}\}$
(d) None of the above

(18) Let X be a non-empty set and $f, g, h : X \longrightarrow X$ be functions. Then, which of the following MAY NOT be true?

(a) $g \circ f(x) = g(f(x))$ for all $x \in X$.
(b) $f \circ g(x) = f(g(x))$ for all $x \in X$.
(c) $f \circ g = g \circ f$
(d) $(f \circ g) \circ h = f \circ (g \circ h)$

(19) Let $f : X \longrightarrow Y$ be a bijective function. Then $g : Y \longrightarrow X$ is said to be the inverse of f

(a) if and only if $f \circ g = id_Y$
(b) if and only if $g \circ f = id_X$
(c) if and only if $f \circ g = id_Y$ and $g \circ f = id_X$
(d) None of these

(20) A function $f : X \longrightarrow Y$ is invertible if and only if

(a) f is injective
(b) f is surjective
(c) f is bijective
(d) None of the above

(21) $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be two bijective functions. The

(a) $(gof)^{-1} = g^{-1}of^{-1}$.
(b) $(gof)^{-1} = f^{-1}og^{-1}$.
(c) $(gof)^{-1} = gof^{-1}$.
(d) $(gof)^{-1} = fog^{-1}$.

(22) Which of the following is NOT TRUE

(a) Inverse of $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = e^x$ is $g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = \log x$.
(b) Inverse of $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x + 3$ is $g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = x - 3$.
(c) Inverse of $f : \mathbb{R}\backslash\{-1\} \longrightarrow \mathbb{R}\backslash\{1\}, f(x) = \frac{x+5}{1+x}$ is $g : \mathbb{R}\backslash\{1\} \longrightarrow \mathbb{R}\backslash\{-1\}, g(x) = \frac{5+x}{x-1}$.
(d) Inverse of $f : [0,1] \longrightarrow [0,1], f(x) = x^2$ is $g : [0,1] \longrightarrow [0,1], g(x) = \sqrt{x}$.

(23) $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be two functions.

(a) gof is injective $\implies$ f, g are injective
(b) gof is surjective $\implies$ f, g are both surjective
(c) gof is bijective $\implies$ f, g are both bijective
(d) f, g are bijective $\implies$ gof is bijective

(24) $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ be two functions.

(a) gof is injective $\implies$ g is injective
(b) gof is surjective $\implies$ f is surjective
(c) gof is injective and f is surjective $\implies$ g is injective
(d) gof is surjective and g is surjective $\implies$ f is surjective

(B) Descriptive Questions

(1) Determine whether following relations are functions from X to Y?
$X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3, y_4, y_5\}$

(a) $R_1 = \{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$
(b) $R_2 = \{(x_1, y_4), (x_2, y_5), (x_4, y_3)\}$
(c) $R_3 = \{(x_1, y_2), (x_1, y_3), (x_2, y_5), (x_3, y_2), (x_4, y_1)\}$
(d) $R_4 = \{(x_1, y_3), (x_2, y_3), (x_3, y_3), (x_4, y_3)\}$

(2) Prove or disprove:

(a) There is an injective function $f : X \longrightarrow Y$ where $|A| = n, |B| = m$ and $m > n$.
(b) There is a surjective function $f : X \longrightarrow Y$ where $|A| = n, |B| = m$ and $m < n$.

(3) Determine whether the following functions are injective or surjective?

(a) $A = \{a, b\}, B = \{1, 2\}$ and $f_1 = \{(a, 1), (b, 1)\}$
(b) $A = \{a, b\}, B = \{1, 2, 3\}$ and $f_2 = \{(a, 3), (b, 1)\}$
(c) $A = \{a, b\}, B = \{1\}$and $f_3 = \{(a, 1), (b, 1)\}$
(d) $A = \{a, b\}, B = \{1, 2\}$ and $f_4 = \{(a, 2), (b, 1)\}$

(4) Give an example of a function f such that f is

(a) injective but not surjective.
(b) surjective but not injective.
(c) injective as well as surjective.
(d) neither injective nor surjective.

(5) Determine whether following functions are injective/ surjective

(a) $f : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$, $f(n) =$ last digit of n for all $n \in \mathbb{N}$.
(b) $f : \mathbb{N} \longrightarrow \mathbb{N}$, $f(n) =$ sum of digits of n for all $n \in \mathbb{N}$.
(c) $f : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$, $f(n) =$ number of digits of n for all $n \in \mathbb{N}$.
(d) $f : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$, $f(n) = n - 1$ for all $n \in \mathbb{N}$.
(e) $f : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$, $f(n) = n + 1$ for all $n \in \mathbb{N}$.
(f) $f : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\}$. $f(n) = \begin{cases} 0 & \text{if n is even,} \\ 1 & \text{if n is odd.} \end{cases}$
(g) $f : \mathbb{Z} \longrightarrow \mathbb{Z}$, $f(k) = 3k$ for all $k \in \mathbb{Z}$.
(h) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x$ for all $x \in \mathbb{R}$.
(i) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^2$ for all $x \in \mathbb{R}$.
(j) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = \sin x$ for all $x \in \mathbb{R}$.
(k) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = e^x$ for all $x \in \mathbb{R}$.
(l) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = \log x$ for all $x \in \mathbb{R}$.
(m) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = |x|$ for all $x \in \mathbb{R}$.

(6) Describe the following functions and write its range.

(a) Identity function.
(b) Inclusion function.
(c) Constant function.
(d) Projection function.
(e) Characteristic function.

(7) Find Range of each of the following function.

(a) $f : X \longrightarrow Y$ where X = {Apple, Banana, Grapes, Cherry} and Y = {A, B, $\cdots$,Z} and f(Apple) = A, f(Banana) = B, f(Grapes) = G, f(Cherry) = C.
(b) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^2 + 2$ for all $x \in \mathbb{R}$.
(c) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = \sin(x - \frac{\pi}{2})$ for all $x \in \mathbb{R}$.

(d) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = |x + 2|$ for all $x \in \mathbb{R}$.
(e) $f : \mathbb{Z} \longrightarrow \mathbb{Z},\ f(k) = 2k$ for all $k \in \mathbb{Z}$.
(f) $f : \mathbb{Z} \longrightarrow \mathbb{Z},\ f(k) =$ remainder when k is divided by 7 for all $k \in \mathbb{Z}$.
(g) $f : \{1, 2, \cdots, 1000\} \longrightarrow \mathbb{N},\ f(n) =$ sum of digits of n for all $n \in \{1, 2, \cdots, 1000\}$.
(h) $f : \mathbb{N} \longrightarrow \mathbb{N} \cup \{0\},\ f(n) =$ last digit of n for all $n \in \mathbb{N}$.
(i) $f : \{1, 2, \cdots, 1000\} \longrightarrow \mathbb{N},\ f(n) =$ number of digits of n for all $n \in \{1, 2, \cdots, 1000\}$.
(j) $f : [-1, 1] \longrightarrow \mathbb{R}, f(x) = \sqrt{(1 - x^2)}$ for all $x \in [-1, 1]$.
(k) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = 3x + 11$ for all $\in \mathbb{R}$.
(l) $f : \mathbb{R}\backslash\{2\} \longrightarrow \mathbb{R}, f(x) = \frac{3}{2-x}$ for all $x \in \mathbb{R}\backslash\{2\}$.
(m) $f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, f(a, b) = (a + b, a - b)$ for all $(a, b) \in \mathbb{R} \times \mathbb{R}$.

(8) For the floor function f find

(a) $f([1, 5])$ (b) $f((-\infty, 0])$ (c) $f(\{0\})$ (d) $f(\{0, 1\})$ (e) $f(\mathbb{N})$

(9) If $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^2$ for all $x \in \mathbb{R}$, then find

(a) $f(\{-1, 1\})$ (b) $f((-1, 1))$ (c) $f([0, 1))$ (d) $f^{-1}([0, 4])$ (e) $f^{-1}(\mathbb{R})$

(10) Let X be a non empty set and $A \subseteq X$ then

(a) Find $\chi_A^{-1}(\{1\})$ and $\chi_A^{-1}(\{0\})$ where χ_A is the characteristic function of X.
(b) For which $A \subseteq X$, $\chi_A^{-1}(\{1\}) = X$, $\chi_A^{-1}(\{1\}) = \emptyset$.

(11) If $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^3 + 3\ \forall\ x \in \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = 2x + 1\ \forall\ x \in \mathbb{R}$, find

(a) $gof(1)$ (b) $fog(1)$ (c) $fof(0)$ (d) $((fog)og)(0)$

(12) Find fog and gof in each of the following case. Further check whether $fog = gof$.

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^2$ and $g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = 2x + 1$.
(b) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = sinx$ and $g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = \lfloor x \rfloor$.
(c) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = e^x$ and $g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = \log x$.
(d) $f : \mathbb{R}\backslash\{1\} \longrightarrow \mathbb{R}\backslash\{0\}, f(x) = \frac{1}{1-x}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}, g(x) = \frac{x-1}{x}$.

(13) Prove that following functions are bijective also find inverse function in each case.

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = 2x + 3\ \forall\ x \in \mathbb{R}$.
(b) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = 5x - 8\ \forall\ x \in \mathbb{R}$.
(c) $f : \mathbb{R}\backslash\{3\} \longrightarrow \mathbb{R}\backslash\{0\}, f(x) = \frac{1}{x-3}\ \forall\ x \in \mathbb{R}\backslash\{3\}$.
(d) $f : \mathbb{R}\backslash\{1\} \longrightarrow \mathbb{R}\backslash\{-1\}, f(x) = \frac{x+5}{1-x}\ \forall\ x \in \mathbb{R}\backslash\{1\}$.
(e) $f : \mathbb{R}\backslash\{3\} \longrightarrow \mathbb{R}\backslash\{5\}, f(x) = \frac{5x+1}{x-3}\ \forall\ x \in \mathbb{R}\backslash\{3\}$.
(f) $f : \mathbb{R}\backslash\{\frac{-1}{2}\} \longrightarrow \mathbb{R}\backslash\{\frac{3}{2}\}, f(x) = \frac{3x+5}{2x+1}\ \forall\ x \in \mathbb{R}\backslash\{\frac{-1}{2}\}$
(g) $f : \mathbb{R}\backslash\{\frac{-3}{5}\} \longrightarrow \mathbb{R}\backslash\{\frac{9}{5}\}, f(x) = \frac{9x+5}{5x+3}\ \forall\ x \in \mathbb{R}\backslash\{\frac{-3}{5}\}$.

2.4

(h) $f : \mathbb{R} \longrightarrow \mathbb{R}^+, f(x) = e^x \ \forall \ x \in \mathbb{R}$.
(i) $f : \mathbb{C} \longrightarrow \mathbb{C}, f(z) = \bar{z} \ \forall \ z \in \mathbb{C}$.

(14) $f : \mathbb{R}\backslash\{1\} \longrightarrow \mathbb{R}\backslash\{0\}, f(x) = \frac{1}{x-1}$, $g : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}\backslash\{0\}, f(x) = \frac{1}{x}$. Find $(gof)^{-1}$.

xxxxxxxxx

2.4 Practical 2.4: Binary Operations, Equivalence Relations, Partition and Equivalence classes

2.4.1 Prerequisite for Practical 2.4

(1) **Binary Operation**: Let X be a non-empty set. Any function $*$ from $X \times X$ to X is called a **binary operation** on X.
Hence if $a, b \in X$ and $*$ is a binary operation on X then $a * b \in X$.
Note: Binary operations are denoted by different notations such as $\Delta, \circ, \cdots$ etc.

(2) Some binary operations are as follows.

(i) Addition is a binary operation on $\mathbb{N}$ (also on $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$) as for any $a, b \in \mathbb{N}$ (or $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$), $a + b \in \mathbb{N}$. (or $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$)
(ii) Multiplication is also a binary operation on $\mathbb{N}$. (also on $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$).

(3) Some non-binary operations:

(i) Subtraction is not a binary operation on $\mathbb{N}$ as $4, 6 \in \mathbb{N}$ but $4 - 6 = -2 \notin \mathbb{N}$,
(ii) Division is not a binary operation on $\mathbb{N}$ as $2, 3 \in \mathbb{N}$ but $\dfrac{2}{3} \notin \mathbb{N}$.

(4) Let X be a non-empty set with a binary operation $*$ on it. Let A be a non-empty subset of X. Then A is said to be **closed** under the binary operation $*$, if $a, b \in A \Longrightarrow a * b \in A$.

(5) Let X be a non-empty set and $*$ be a binary operation on it.

(i) $*$ is said to be **commutative**, if $a * b = b * a$ for all $a, b \in X$.
(ii) $*$ is said to be **associative**, if $a * (b * c) = (a * b) * c$ for all $a, b, c \in X$.
(iii) If there exists an element e in X such that $x * e = x = e * x$, for all $x \in X$, then, e is said to be an **identity element** of X with respect to the binary operation $*$.
(iv) If e is an identity element with respect to the binary operation $*$ and for an element $a \in X$, there exists an element $b \in X$, such that $a * b = e = b * a$, then it is called as an **inverse** of a with respect to $*$.

(6) **Relations**:
If X and Y are two non-empty sets then a **relation** R from X to Y is a subset of $X \times Y$.(prerequisite for Practical 2.4 number (3),)
If $(a, b) \in R$ then we write $a \ R \ b$ and read as a is $R-$related to b or a is related to b under R.

Remarks 2.4.1

(i) If X or Y is an empty set then $X \times Y$ is also an empty set and any relation defined on these sets is also empty.

(ii) R can still be $\emptyset$ even when X and Y are non-empty.

(iii) We are going to consider $X \neq \emptyset, Y \neq \emptyset$ and $R \subseteq X \times Y, R \neq \emptyset$.

(iv) If $X = Y$, that is if R is a relation from X to itself, then we say that R is a relation on X.

(7) Let X be a non-empty set and let R be a relation on it. Then R is said to be

(i) **Reflexive**, if $a\ R\ a$, for all $a \in X$.

(ii) **Symmetric**, if $a\ R\ b \Longrightarrow b\ R\ a$, for all $a, b \in X$.

(iii) **Transitive,** if $a\ R\ b, b\ R\ c \Longrightarrow a\ R\ c$, for all $a, b, c \in X$.

(8) **Equivalence Relation**: Let X be a non-empty set and let R be a relation on X. Then R is said to be an **equivalence relation** on X if R is reflexive, symmetric and transitive. An equivalence relation R is denoted by $\tilde{}$(tilda).

(9) Some equivalence relations:

(i) Let X be a non-empty set. Define a relation R on X as aRb if and only if $a = b$ for all $a, b \in X$. Then we can verify that R is an equivalence relation on X. This equivalence relation is called as a **trivial equivalence relation**.

(ii) Define a relation R on $\mathbb{Z}$ set of integers as follows: For $a, b \in \mathbb{Z}, a\ R\ b$ if and only if $a + b$ is an even integer. We can verify that R is an equivalence relation.

(iii) Let n be a positive integer. Define a relation R on $\mathbb{Z}$ such that $a\ R\ b$ if and only if $a \equiv b(\mod n)$ for all $a, b \in \mathbb{Z}$. Then R is an equivalence relation on $\mathbb{Z}$.

(10) **Equivalence Class**: Let X be a non-empty set and let R be an equivalence relation on X. If $a \in X$, then the set $\{x \in X : x\ R\ a\}$ is called the **equivalence class** of a with respect to the equivalence relation R. The equivalence class of a is denoted by $[a]$ or $\overline{a}$.

(11) **Partition of a set**: Let X be a non-empty set. $P = \{A_1, A_2, \cdots\}$ is a **partition** of X if the following conditions are satisfied.
(1) Each A_i is a non-empty subset of X.
(2) $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i, A_j \in P$.
(3) $\cup_{A_i \in P} A_i = X$.

(12) Let X be a non-empty set and let R be an equivalence relation on X. Let P be the set of all equivalence classes of elements of X with respect the relation R. That is, $P = \{[a] : a \in X\}$ where $[a] = \{x \in X : x\ R\ a\}$. Then P is a partition of X.

(13) Let X be a non-empty set and let $P = \{A_1, A_2, \cdots\}$ be a partition of X. Define a relation R on X such that for $a, b \in X, a\ R\ b$ if and only if a and b belong to the same A_i for some $A_i \in P$. Then R is an equivalence relation on X.

(14) Let X be a non-empty set. An equivalence relation on X induces a partition on X and conversely, every partition of X defines an equivalence relation on X.

2.4

(15) **Equivalence Classes modulo n:**
Let n be a positive integer. Define a relation R on $\mathbb{Z}$ such that $a\ R\ b$ if and only if $a \equiv b($ mod $n)$. Then R is an equivalence relation on $\mathbb{Z}$. Hence R induces a partition on Z containing all equivalence classes of all elements of $\mathbb{Z}$ with respect to the relation R. This partition is denoted by $\mathbb{Z}_n$.
Hence $\mathbb{Z}_n = \{\overline{a} : a \in \mathbb{Z}\}$ where $\overline{a} = \{x \in \mathbb{Z} : xRa\}$. This partition is denoted by $\mathbb{Z}_n$ and called as the set of all **residue classes modulo** n.
Thus $\mathbb{Z}_n = \{\overline{a} : a \in \mathbb{Z}\}$.

$$\begin{aligned}
\overline{a} &= \{x \in \mathbb{Z} : x\ R\ a\}.\\
&= \{x \in \mathbb{Z} : n \mid (x-a)\}.\\
&= \{x \in \mathbb{Z} : x - a = nk \text{ for some } k \in \mathbb{Z}\}.\\
&= \{x \in \mathbb{Z} : x = a + nk \text{ for some } k \in \mathbb{Z}\}.\\
&= \{a + nk : k \in \mathbb{Z}\}. \quad (1)
\end{aligned}$$

By Division Algorithm, $a = nq + r, 0 \leq r < n$. We can show that $\overline{a} = \overline{r}$.
Since the remainders can take values only from 0 to $n-1$, the only distinct residue classes modulo n are $\overline{0}, \overline{1}, \cdots, \overline{n-1}$. Hence $\mathbb{Z}_n = \{\overline{0}, \overline{1}, \cdots, \overline{n-1}\}$.

(16) $\mathbb{Z}_5 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}\}$. From (1), we have,

$$\begin{aligned}
\overline{0} &= \{0 + 5k : k \in Z\} = \{5k : k \in \mathbb{Z}\} = \{\cdots, -20, -15, -10, -5, 0, 5, 10, \cdots\}\\
\overline{1} &= \{1 + 5k : k \in Z\} = \{\cdots, -19, -14, -9, -4, 1, 6, 11, \cdots\}\\
\overline{2} &= \{2 + 5k : k \in Z\} = \{\cdots, -18, -13, -8, -3, 2, 7, 12, \cdots\}\\
\overline{3} &= \{3 + 5k : k \in Z\} = \{\cdots, -17, -12, -7, -2, 3, 8, 13, \cdots\}\\
\overline{4} &= \{4 + 5k : k \in Z\} = \{\cdots, -16, -11, -6, -1, 4, 9, 14, \cdots\}
\end{aligned}$$

(17) If n is a positive integer and $a, b \in \mathbb{Z}$ then following statements are true for $\mathbb{Z}_n$.

(i) $a \in \overline{a}$.
(ii) $a \equiv b \mod n$ if and only if $\overline{a} = \overline{b}$.
(iii) If $a = nq + r, 0 \leq r < n$ then $\overline{a} = \overline{r}$.

(18) Binary operations on $\mathbb{Z}_n$ where n is a positive integer.
Let $a, b \in \mathbb{Z}$. Define $+$ and $*$ on $\mathbb{Z}_n$ as follows:

(i) $\overline{a} + \overline{b} = \overline{a+b}$.
(ii) $\overline{a} * \overline{b} = \overline{a * b}$.

We will prepare the addition table and multiplication table for $\mathbb{Z}_6$.

$+$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{1}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{0}$
$\overline{2}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{1}$
$\overline{3}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$
$\overline{4}$	$\overline{4}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
$\overline{5}$	$\overline{5}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$

$*$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$	$\overline{0}$
$\overline{1}$	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$
$\overline{2}$	$\overline{0}$	$\overline{2}$	$\overline{4}$	$\overline{0}$	$\overline{2}$	$\overline{4}$
$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$	$\overline{0}$	$\overline{3}$
$\overline{4}$	$\overline{0}$	$\overline{4}$	$\overline{2}$	$\overline{0}$	$\overline{4}$	$\overline{2}$
$\overline{5}$	$\overline{0}$	$\overline{5}$	$\overline{4}$	$\overline{3}$	$\overline{2}$	$\overline{1}$

2.4.2 PRACRICAL 2.4

(A) Objective Questions:

Choose correct alternative in each of the following:

(1) If $*$ is a binary operation on $\mathbb{N}$ then $*$ can be

(a) Addition (b) Subtraction (c) Division (d) None of these

(2) If $*$ is an associative binary operation on $\mathbb{Z}$ then $*$ can be

(a) Addition (b) Subtraction (c) Division (d) None of these

(3) Which of the following statement is false?

(a) Addition is a binary operation on $\mathbb{N}$.
(b) Addition is a binary operation on $\mathbb{Z}$.
(c) Addition is a binary operation on $\mathbb{R}^+$.
(d) All above statements are false.

(4) Division is a binary operation on

(a) $\mathbb{Z}$ (b) $\mathbb{Q}$ (c) $\mathbb{R}$ (d) $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$

(5) Subtraction is not a binary operation on

(a) $\mathbb{N}$ (b) $\mathbb{Z}$ (c) $\mathbb{Q}$ (d) $\mathbb{R}$

(6) In which of the following sets, every element has multiplicative inverse?

(a) $\mathbb{Z}$ (b) $\mathbb{Q}$ (c) $\mathbb{R}$ (d) $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$

(7) Consider the binary operation $*$ on $\mathbb{Z}$ as follows. For, $a, b \in \mathbb{Z}$, $a * b = a + b - 7$. Identity element of $\mathbb{Z}$ under the binary operation $*$ is?

(a) 0 (b) 1 (c) 7 (d) -7

(8) Consider the binary operation $*$ on $\mathbb{Q}^* = \mathbb{Q}\backslash\{0\}$ as follows. For, $a, b \in \mathbb{Q}^*$, $a * b = \frac{ab}{3}$. Identity element of $\mathbb{Q}^*$ under the binary operation $*$ is?

(a) 0 (b) 1 (c) 3 (d) $\frac{1}{3}$

(9) For $a, b \in \mathbb{R}$, $a * b = |a|b$. Then which of the following is NOT TRUE?

(a) $*$ is a binary operation on $\mathbb{R}$.
(b) $*$ is commutative.
(c) $*$ is associative.
(d) All of these.

(10) Which of the following binary operation on $\mathbb{Q}\backslash\{0\}$ is not commutative?

(a) $a * b = 4a - 5b$.
(b) $a * b = a + b - 5$.
(c) $a * b = \frac{ab}{5}$.
(d) $a * b = a^2 + b^2$.

(11) For $a, b \in \mathbb{N}$, $a * b = max\{a, b\}$. Then, what is NOT TRUE ?

(a) $*$ is a binary operation on $\mathbb{R}$.
(b) $*$ is commutative.
(c) Identity element of $\mathbb{N}$ under the operation $*$ is 1.
(d) Every element of $\mathbb{N}$ has inverse element in $\mathbb{N}$ under the binary operation $*$.

(12) The set of all integers having multiplicative inverse is

(a) $\{1\}$. (b) $\{-1\}$. (c) $\{-1, 0, 1\}$. (d) $\{-1, 1\}$.

(13) For $A, B \in M_n(\mathbb{R})$, $A * B = A + B$. Then,

(a) $*$ is not a binary operation on $M_n(\mathbb{R})$.
(b) $*$ is a binary operation on $M_n(\mathbb{R})$ but not commutative.
(c) $*$ is a binary operation on $M_n(\mathbb{R})$ and is commutative too.
(d) None of these.

(14) $X = \left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix} \mid a \neq 0, a \in \mathbb{Q}\right\}$. Consider the operation $*$ on X as follows, for $A, B \in X$, $A * B = AB$. Then, which of the following is NOT TRUE?

(a) $*$ is a binary operation on X.
(b) $*$ is a commutative binary operation on X.
(c) X doesn't have an identity element under the binary operation $*$.
(d) Every matrix in X has an inverse element in under the binary operation $*$.

(15) Let X be a non empty set and $S = \mathfrak{P}(X)$. The identity element of S under the binary operation $'\cup'$(Union) is

(a) X. (b) $\emptyset$. (c) X^c. (d) None of these.

(16) Let X be a non empty set and $S = \mathfrak{P}(X)$. The element of S having inverse in S under the binary operation $'\cap'$ (Intersection) is

(a) X. (b) $\emptyset$. (c) X^c. (d) None of these.

(17) Let X be a non empty set and $S = \{f : X \longrightarrow X | f \text{is bijective}\}$. For, $f, g \in S, f * g = f \circ g$. Then, the identity element of under the binary operation $*$ is

(a) $f : X \longrightarrow X, f(x) = x$ for all $x \in X$.
(b) $f : X \longrightarrow X, f(x) = 0$ for all $x \in X$.
(c) $f : X \longrightarrow X, f(x) = 1$ for all $x \in X$.
(d) None of these.

(18) Identity element of $\mathbb{Z} \times \mathbb{Z}$ under the operation $*$ defined as for $(a, b), (a', b') \in \mathbb{Z} \times \mathbb{Z}$, $(a, b) * (a', b') = (aa', bb')$

(a) $(1, 0)$. (b) $(0, 1)$. (c) $(1, 1)$. (d) $(0, 0)$.

(19) The elements of $\mathbb{Z} \times \mathbb{Z}$ having an inverse in $\mathbb{Z} \times \mathbb{Z}$ under the operation $*$ defined as for $(a, b), (a', b') \in \mathbb{Z} \times \mathbb{Z}$, $(a, b) * (a', b') = (aa', bb')$ are

(a) $\{(1, 0)\}$.
(b) $\{(1, 1), (-1, 0)\}$.
(c) $\{(0, -1), (-1, 1)\}$.
(d) $\{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$.

(20) $X = \{a, b, c, d\}, R = \{(a, a), (a, b), (b, d), (c, c), (b, a)\}$, then

(a) R is reflexive.
(b) R is symmetric.
(c) R is transitive.
(d) None of these.

(21) $X = \{1, 2, 3, 4\}, R = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1)\}$, then

(a) R is reflexive.
(b) R is symmetric.
(c) R is transitive.
(d) None of these.

(22) For $a, b \in \mathbb{N}$, $aRb \iff a + b$ is odd. Then R is

(a) reflexive but not symmetric
(b) Not reflexive but symmetric
(c) Only symmetric
(d) Neither reflexive nor symmetric

(23) For $x, y \in \mathbb{R}$, $xRy \iff$ there exists $c \neq 0, c \in \mathbb{R}$ such that $y = cx$ then

(a) R is not an equivalence relation
(b) R is an equivalence relation with two distinct equivalence classes
(c) R is an equivalence relation with exactly one equivalence class
(d) R is an equivalence relation with infinitely many distinct equivalence classes

(24) Number of different relations on a non empty set X containing 'n' elements is

2.4

(a) 2^{n^2} (b) 2^n (c) n^2 (d) None of these

(25) Number of different reflexive relations on a non empty set X containing 'n' elements is

(a) 2^{n^2} (b) $2^{n(n-1)}$ (c) $2^{n^2(n-1)}$ (d) $2^{\frac{n(n-1)}{2}}$

(26) Number of different symmetric relations on a non empty set X containing 'n' elements is

(a) $2^{\frac{n^2+2n-1}{2}}$ (b) 2^{n^2+2n-1} (c) $2^{\frac{n^2+n-1}{2}}$ (d) None of these

(27) Which of the following statements is NOT TRUE

(a) Every equivalence relation induces a partition and vice versa
(b) Two equivalence classes are either equal or disjoint
(c) Intersection of two equivalence relations is again an equivalence relation.
(d) Union of two equivalence relations is again an equivalence relation.

(28) $X = \{a, b, c\}, R = \{(a, a), (b, b), (c, c), (a, b), (b, a)\}$. The partition induced by the equivalence relation R on X is

(a) $\{\{a\}, \{b\}, \{c\}\}$ (c) $\{\{a\}, \{b, c\}\}$
(b) $\{\{a, b\}, \{c\}\}$ (d) $\{\{a, b, c\}\}$

(29) $X = \{1, 2, 3\}, P = \{\{1, 3\}, \{2\}\}$. The equivalence relation induced by the partition P is

(a) $\{(1, 1), (3, 3), (1, 3), (3, 1), (2, 2)\}$ (c) $\{(1, 1), (2, 2), (3, 3)\}$
(b) $\{(1, 1), (2, 2), (1, 2), (2, 1), (3, 3)\}$ (d) None of these

(30) For $a, b \in \mathbb{Z}$, $aRb \iff a \equiv b(mod\ 5)$. Then the number of distinct equivalence classes are

(a) 0 (b) 4 (c) 6 (d) 5

(31) For $a, b \in \mathbb{Z}$, $aRb \iff a \equiv b(mod\ n)$ (n is a positive integer). Then R is

(a) Reflexive (b) Symmetric (c) Transitive (d) All the above

(32) In $\mathbb{Z}_{62}, \overline{31} + \overline{35}$ is

(a) $\overline{4}$ (b) $\overline{14}$ (c) $\overline{20}$ (d) None of these

(33) In $\mathbb{Z}_{14}, \overline{35} - \overline{39}$ is

(a) $\overline{4}$ (b) $\overline{10}$ (c) $\overline{20}$ (d) None of these

(34) In $\mathbb{Z}_{62}, \overline{31} * \overline{35}$ is

(a) $\overline{31}$ (b) $\overline{14}$ (c) $\overline{25}$ (d) None of these

(35) In $\mathbb{Z}_{14}, \overline{-35} * \overline{39}$ is

(a) $\overline{4}$ (b) $\overline{7}$ (c) $\overline{27}$ (d) None of these

(B) Descriptive Questions

(1) Check whether the given operation $*$ is a binary operation on the given set S. If $*$ is a Binary operation, then find the identity element of S(if it exists). Further, if S has an identity element under the binary operation $*$, then determine those elements of S that have an inverse in S.

(i) $S = \mathbb{N}$, for $a, b \in \mathbb{N}$, $a * b = a + b$.
(ii) $S = \mathbb{N}$, for $a, b \in \mathbb{N}$, $a * b = max\{a, b\}$.
(iii) $S = \mathbb{Z}$, for $a, b \in \mathbb{Z}$, $a * b = a - b$.
(iv) $S = \mathbb{Z}$, for $a, b \in \mathbb{Z}$, $a * b = a + b - 3$.
(v) $S = \mathbb{Q}^*$, for $a, b \in \mathbb{Q}^*$, $a * b = \frac{ab}{5}$.
(vi) $S = \mathbb{R} \setminus \{1\}$, for $a, b \in \mathbb{R} \setminus \{1\}$, $a * b = a + b - ab$.
(vii) $S = \mathbb{R}$, for $a, b \in \mathbb{R}$, $a * b = |a - b|$.
(viii) $S = M_n(\mathbb{R})$, for $A, B \in M_n(\mathbb{R})$, $A * B = A + B$.
(ix) $S = M_n(\mathbb{R})$, for $A, B \in M_n(\mathbb{R})$, $A * B = AB$.
(x) $S = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} |a \neq 0, a \in \mathbb{Q} \right\}$, for $A, B \in S$, $A * B = AB$.
(xi) $S = \mathbb{Z} \times \mathbb{Z}$, for $(a, b), (a', b') \in \mathbb{Z} \times \mathbb{Z}$, $(a, b) * (a', b') = (aa', bb')$.

(2) Let X be a non empty set and $S = \mathfrak{P}(X)$. Is $*$ a binary operation on S? Find the identity element of S (if it exists), and invertible elements of S.

(i) For $A, B \in S, A * B = A \cup B$.
(ii) For $A, B \in S, A * B = A \cap B$.
(iii) For $A, B \in S, A * B = A \setminus B$.

(3) Let X be a non empty set and $S = \{f : X \longrightarrow X | f \text{ is bijective}\}$. For, $f, g \in S, f * g = f \circ g$. Is $*$ a binary operation on S? Find the identity element of S (if it exists), and invertible elements of S. Is $*$ associative? Is it commutative?

(4) Determine whether following relation R on set X is reflexive, symmetric and transitive and hence an equivalence relation. If R is an equivalence relation then find all its equivalence classes.

(i) $X = \{1, 2, 3, 4\}, R = \{(1, 2), (1, 3), (3, 1), (1, 1), (3, 3), (3, 2), (1, 4), (4, 2), (3, 4)\}$.
(ii) $X = \{a, b, c, d, e\}, R = \{(a, a), (a, b), (a, c), (b, a), (b, b), (c, a), (b, c), (c, c), (d, d), (c, b), (e, e)\}$.

(5) Below is the list of relations among people. For each of the relations, state whether the relation is reflexive, symmetric or transitive. Let $X :=$ The set of all people

(i) For $x, y \in X$, $xRy \iff$ Age of x and y is same.

(ii) For $x, y \in X$, $xRy \iff x$ is taller than y.

(iii) For $x, y \in X$, $xRy \iff x$ is a relative of y.

(6) Suppose in a party there are n guests and n identical chairs placed around table and S is the set of all possible sitting arrangements. Define a relation R as: For $A_1, A_2 \in S, A_1RA_2 \iff A_2$ can be obtained from A_1 by each guest moving a fixed number 'r' places in a clockwise direction. Determine whether the relation R defined as above is an equivalence relation or not?

(7) Let A be the set of all strings that contains English alphabets. Define a relation R as: For each, $a, b \in A, aRb \iff l(a) = l(b)$ where $l(x) :=$ length of string x. Determine whether relation R defined as above is a equivalence relation or not?

(8) Check whether following relations are equivalence relations. If Yes, then describe their distinct equivalence classes.

(i) For $a, b \in \mathbb{N}$, $aRb \iff a + b$ is odd.

(ii) For $a, b \in \mathbb{Z}$, $aRb \iff a - b$ is odd.

(iii) For $a, b \in \mathbb{Z}$, $aRb \iff a \leq b$.

(iv) For $x, y \in \mathbb{Z}$, $xRy \iff x - y$ is divisible by 4.

(v) For $x, y \in \mathbb{Z}$, $xRy \iff 2x + y$ is divisible by 3.

(vi) For $x, y \in \mathbb{Z}$, $xRy \iff 3x + 7y$ is divisible by 10.

(vii) $X = \mathbb{R} \times \mathbb{R}$. For $(x_1, y_1), (x_2, y_2) \in X, (x_1, y_1)R(x_2, y_2) \iff x_1 + x_2 = y_1 + y_2$.

(viii) $X = \mathbb{Z} \times \mathbb{Z}$. For $(a, b), (c, d) \in X, (a, b)R(c, d) \implies ad = bc$.

(9) X = Set of all Straight lines in plane $\mathbb{R}^2$. Check whether following relations are equivalence. If yes, describe distinct equivalence classes.

(i) For $l_1, l_2 \in X$, $l_1Rl_2 \iff l_1$ is parallel to l_2.

(ii) For $l_1, l_2 \in X$, $l_1Rl_2 \iff l_1$ is perpendicular to l_2.

(iii) For $l_1, l_2 \in X$, $l_1Rl_2 \iff l_1$ is either parallel or perpendicular to l_2.

(10) Let S be a non empty set and $X = \mathfrak{P}(S)$. Check whether following relations are equivalence. If yes, describe distinct equivalence classes.

(i) For $A, B \in X$, $ARB \iff A \subseteq B$.

(ii) For $A, B \in X$, $ARB \iff |A| = |B|$.

(11) X = Set of all functions from $\mathbb{Z}$ to $\mathbb{Z}$. Check whether following relations are equivalence. If yes, describe the distinct equivalence classes

(i) For $f, g \in X$, $fRg \iff f(0) = g(0)$.

(ii) For $f, g \in X$, $fRg \iff f(0) = g(1)$ or $g(0) = f(1)$.

(12) How many different relations, reflexive and symmetric can be defined on a set X containing n elements.

(13) Show that if R and S are equivalence on set X then $R \cap S$ is also an equivalence relation on X.

(14) Let R be an equivalence relation on X. Let $x \in X$ then prove that $y \in [x] \iff [x] = [y]$.

(15) $X = \{1, 2, 3, \cdots, 10\}$. Find any three partitions of X containing 5 parts.

(16) List down any 5 partitions of $X = \{a, b, c, d, e\}$.

(17) $X = \{1, 2, 3, 4, 5, 6\}$, $R = \{(1, 1), (1, 5), (2, 2), (2, 3), (2, 6), (3, 2), (3, 3), (3, 6), (4, 4), (5, 1), (5, 5), (6, 2), ($
Find the partition of X induced by R.

(18) Let relation R be defined on $A = \{1, 2, 3\}$ as $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 3)\}$. Find the partition on A using equivalence classes.

(19) If $P = \{\{a, c, e\}, \{b, d\}\}$ is a partition of the set $X = \{a, b, c, d, e\}$. Determine the equivalence relation that induces above partitions.

(20) Let $A = \{1, 2, 3, 4, 5, 6\}$ and $P = \{\{1\}, \{3, 6\}, \{2, 4, 5\}\}$. Write the equivalence relation that induces the above partition.

(21) Let $A = \{1, 2, 3, 4\}$ and $P = \{\{1, 2, 3\}, \{4\}\}$ be a partition of A. Find equivalence relation R on A determined by the partition P.

(22) If $A = \{a, b, c\}$. How many relations can be defined on X ? How many of them are equivalence relations? List all the equivalence relations on X.

(23) For $a, b \in \mathbb{Z}$, $aRb \iff a \equiv b(mod\ 7)$.Prove that R is an equivalence relation on $\mathbb{Z}$. Further find all distinct equivalence classes.

(24) Prepare the multiplication table for $\mathbb{Z}_8$.

(25) Prepare the addition table for $\mathbb{Z}_9$.

xxxxxxxxx

2.5 Practical 2.5: Polynomials (I)

2.5.1 Prerequisite for Practical 2.5

(1) **Polynomial**: Let F denote $\mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. A polynomial over F is an expression in x of the form $c_0 + c_1x + \cdots + c_nx^n$ where n is a nonnegative integer and $c_0, c_1, \cdots, c_n \in F$.
$c_0, c_1, \cdots, c_n$ are called the **coefficients** of the above polynomial and x is called the indeterminate.
c_i is called the coefficient of x^i, for $i = 0, 1, \cdots, n$.

(2) If $c_n \neq 0$, the polynomial is said to have **degree** n, and c_n is called the **leading coefficient**.

(3) A polynomial whose leading coefficient is 1 is said to be **monic** polynomial.

(4) Two polynomials $a_0 + a_1x + \cdots + a_nx^n$ and $b_0 + b_1x + \cdots + b_mx^m$ are said to be equal if $n = m$ and the corresponding coefficients are equal, that is, $a_0 = b_0, a_1 = b_1, \cdots, a_n = b_m$.

(5) $c_nx^n + \cdots + c_1x + c_0$ is the **zero polynomial** if and only if $c_0 = c_1 = \cdots = c_n = 0$. The degree of the zero polynomial is not defined.

(6) If $p(x)$ is a nonzero polynomial, then its degree is denoted by $\deg p(x)$.

(7) Polynomials of degrees 1, 2 and 3 are called as **linear**, **quadratic**, and **cubic** polynomials, respectively.

(8) Polynomials of degree zero as well as the zero polynomial are called **constant polynomial**.

(9) The set of all polynomials in x with coefficients in F is denoted by $F[x]$ where ($F = \mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$.)
Hence, The set of all polynomials in x with coefficients in $\mathbb{Q}$ is denoted by $\mathbb{Q}[x]$. Similarly, The set of all polynomials in x with coefficients in $\mathbb{R}$ is denoted by $\mathbb{R}[x]$. And The set of all polynomials in x with coefficients in $\mathbb{C}$ is denoted by $\mathbb{C}[x]$.

(10) **Algebra of Polynomials**:

(A) **Addition**
If $f(x), g(x) \in F[x], f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$, then $f(x) + g(x)$ is given by

$$\begin{aligned} f(x) + g(x) =&(a_0 + b_0) + (a_1 + b_1)x \cdots + (a_n + b_n)x^n && \text{if } n = m,\\ &(a_0 + b_0) + (a_1 + b_1)x \cdots + (a_n + b_n)x^n + b_{n+1}x^{n+1} + \cdots + b_mx^m && \text{if } n < m,\\ &(a_0 + b_0) + (a_1 + b_1)x \cdots + (a_m + b_m)x^n + a_{m+1}x^{m+1} + \cdots + a_nx^n && \text{if } n > m. \end{aligned}$$

Let $f(x), g(x), h(x) \in F[x]$. Then following properties hold.

(i) Addition is a binary operation in $F[x]$. That is, $f(x) + g(x) \in F[x]$.
(ii) Addition is associative: $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$.
(iii) Addition is commutative: $(f(x) + g(x)) = g(x) + f(x)$.
(iv) The zero polynomial is the identity element.
(v) For $f(x) = a_0 + a_1x + \cdots + a_nx^n$, define $(-f)(x) = -a_0 - a_1x - \cdots - a_nx^n$. Then $-f \in F(x)$ and $f(x) + (-f(x)) = 0$.
So, $-f(x)$ is the inverse of $f(x)$ in $F[x]$.
(vi) If $f(x), g(x) \in F[x]$ and $f(x)+g(x) \neq 0$, then $\deg(f(x)+g(x)) \leq \max\{\deg f(x), \deg g(x)\}$.

(B) **Multiplication**: If $f(x), g(x) \in F[x], f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$, then $f(x) \cdot g(x)$ is given by

$$\begin{aligned} f(x) \cdot g(x) &= a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + \cdots + a_nb_mx^{n+m}.\\ &= \sum_{i=1}^{n+m} \Big(\sum_{r+s=i} a_rb_s \Big) x^i. \end{aligned}$$

If we denote $c_i = \sum_{r+s=i} a_r b_s$, for all $i, 0 \leq i \leq n+m$, then

$$\begin{aligned} c_0 &= a_0 b_0 \\ c_1 &= a_0 b_1 + a_1 b_0 \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\ c_i &= a_0 b_i + a_1 b_{i-1} + a_2 b_{i-2} + \cdots + a_i b_0 \end{aligned}$$

Let $f(x), g(x), h(x) \in F[x]$. Then following properties hold.

(i) Multiplication is a binary operation in $F[x]$. That is, $f(x) \cdot g(x) \in F[x]$.

(ii) Multiplication is associative: $(f(x) \cdot g(x)) \circ h(x) = f(x) \cdot (g(x) \cdot h(x))$.

(iii) Multiplication is commutative: $(f(x) \cdot g(x)) = g(x) \cdot f(x)$.

(iv) The polynomial defined as $i(x) = 1$ is the identity polynomial with respect to multiplication.

(v) The multiplicative inverse of any polynomial with degree greater than or equal to 1 does not exist in $F[x]$.

(vi) If $f(x), g(x) \in F[x]$ and $f(x) \cdot g(x) \neq 0$, then $\deg(f(x) \cdot g(x)) = \deg f(x) + \deg g(x)$.

(vii) If $f(x), g(x) \in F[x]$ and $f(x) \neq 0, g(x) \neq 0$ then $f(x) \cdot g(x) \neq 0$. (Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ and $g(x) = b_0 + b_1 x + \cdots + b_m x^m$ where $a_n \neq 0$ and $b_m \neq 0$. As $a_n, b_m \in F$ and $a_n \neq 0, b_m \neq 0$, we have $a_n b_m \neq 0$. Now, $a_n b_m$ is the leading coefficient of $f(x) \cdot g(x)$. Hence $f(x) \cdot g(x) \neq 0$.)

(11) Multiplication is distributive over addition in $F[x]$. That is, $f(x) \cdot (g(x) + h(x)) = f(x) \cdot g(x) + f(x) \cdot h(x)$

(12) The cancellation laws hold in $F[x]$. That is, if $f(x) \neq 0$ and $f(x) \cdot g(x) = f(x) \cdot h(x)$ then $g(x) = h(x)$.

(13) **Division Algorithm in** $F[x]$: Let $f(x)$ and $g(x)$ be two polynomials in $F[x]$ with $g(x) \neq 0$. Then there exist unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x) \cdot q(x) + r(x)$ and either $r(x) = 0$ or $\deg r(x) < \deg g(x)$.

(14) **Divisibility in** $F[x]$: Let $f(x)$ and $g(x)$ be two polynomials in $F[x]$ with $g(x) \neq 0$. We say that $g(x)$ **divides** $f(x)$ in $F[x]$ and write $g(x) \mid f(x)$ if there exists $h(x)$ in $F[x]$ such that $f(x) = g(x)h(x)$. In this case, we also call $g(x)$ a **factor** of $f(x)$.

(15) Let $f(x), g(x)$ be nonzero polynomials in $F[x]$. If $f(x) \mid g(x)$ and $g(x) \mid f(x)$ then $f(x) = k \cdot g(x)$ for some $k \in F$.
Since $f(x)|g(x)$ there exists $h(x) \in F[x]$ such that $g(x) = f(x)h(x)$ $\quad (I)$.
Since $g(x)|f(x)$ there exists $q(x) \in F[x]$ such that $f(x) = g(x)q(x)$ $\quad (II)$.
Putting (I) in (II), we get,

$$\begin{aligned} f(x) &= g(x)q(x) \\ &= f(x)h(x)q(x) \\ f(x)(1 - h(x)q(x)) &= 0 \end{aligned}$$

By cancellation law in $F(x)$ (as given in statement no. (12) above) , $1 - h(x)q(x) = 0$ (as $f(x) \neq 0$).

Hence $h(x)q(x) = 1$. This implies $\deg h(x) + \deg q(x) = \deg 1$. That is, $\deg h(x) + \deg q(x) = 0$.
But $\deg h(x) \geq 0, \deg q(x) \geq 0$. Hence, $\deg h(x) = 0$ and $\deg q(x) = 0$.
Therefore $h(x)$ and $q(x)$ are nonzero constant polynomials.
Let $q(x) = k$ for some $k \in F$.
Hence $f(x) = kg(x)$ for some $k \in F$.

(16) **Remainder Theorem**: Let $f(x) \in F[x]$ and $a \in F$. If $f(x)$ is divided by $(x - a)$ then the remainder is $f(a)$.

(17) **Factor Theorem**: Let $f(x) \in F[x]$ and $a \in F$. Then $f(x)$ is divisible by $(x - a)$ if and only if $f(a) = 0$.

(18) A polynomial of degree n over F has at most n zeros, counting multiplicity.

(19) **Greatest Common Divisior in** $F[x]$: Let $f(x), g(x) \in F[x]$. A polynomial $d(x) \in F[x]$ is said to be a **greatest common divisor** (gcd) of $f(x)$ and $g(x)$ if the following conditions are satisfied.

  (1) $d(x) \mid f(x)$ and $d(x) \mid g(x)$.
  (2) If $h(x) \in F[x]$ is such that $h(x) \mid f(x)$ and $h(x) \mid g(x)$ then $h(x) \mid d(x)$.

Note: If $d(x)$ and $d'(x)$ are two gcds of $f(x)$ and $g(x)$ then $d(x) \mid d'(x)$ and $d'(x) \mid d(x)$. Hence $d(x) = kd'(x)$ for some $k \in F$ (from statement no (15) above). So, if we want the unique gcd then we find the monic gcd.
For example, (1) if $gcd(f(x), g(x)) = 2x^2 + 3$ then $x^2 + \frac{3}{2}$ is also a gcd and it is monic.
(2) If $\gcd(f(x), g(x)) = k, k \neq 0$ then $\gcd(f(x), g(x)) = 1$ as $1 \mid k$ and $k \mid 1$ ($1 = k * \frac{1}{k} \implies k \mid 1$).

(20) **Euclidean Algorithm**: Two non-zero polynomials $f(x)$ and $g(x)$ in $F[x]$ have a greatest common divisor in $F[x]$.

(A) Objective Questions

Choose correct alternative in each of the following:

(1) Monic polynomial is one whose.............

  (a) degree is one.
  (b) leading coefficient is one.
  (c) constant term is one.
  (d) all the coefficients are one.

(2) Degree of a non-zero constant polynomial is ....

  (a) 1 (b) 0 (c) 2 (d) Not defined.

(3) If $f(x)$ and $g(x)$ are two polynomials over $\mathbb{R}$ with degree m and n respectively and $m > n$ then $\deg(f(x) + g(x)) =$ ....

(a) $m+n$. (b) $m*n$. (c) m. (d) n.

(4) If $f(x)$ and $g(x)$ are two nonzero polynomials over $\mathbb{R}$ with degree k and l respectively then $\deg(f(x)\cdot g(x)) =$ ....

(a) $k+l$. (b) $k*l$. (c) k. (d) l.

(5) If $f(x)$ and $g(x)$ are any two polynomials with $\deg(f(x)) = 6$ and $\deg(g(x)) = 5$ then $\deg(f(x) - g(x)) =$

(a) 6. (b) 11. (c) 5. (d) -6.

(6) If $f(x) = x^4 + 2x^2 - 5x + 1$ and $\deg(f(x) + g(x)) = 7$ then $\deg(g(x))$ is

(a) 4 (b) 7 (c) 3 (d) 2

(7) What is the quotient when the polynomial $x^3 - 7x^2 - 9x + 63$ is divided by the polynomial $(x+3)$?

(a) $x^2 - 10x + 21$ (c) $x^2 + 10x - 21$
(b) $x^2 - 10x - 21$ (d) $x^2 + 10x + 21$

(8) If $f(x)$ is divided by $(x - 97)$ then remainder is

(a) 0 (b) 97 (c) 1 (d) $f(97)$

(9) What is the quotient when $x^4 - 3x^2 + 4x + 8$ is divided by $x^2 + 2$?

(a) $x^2 - \frac{1}{2}$ (b) $x^2 - 5$ (c) $x^2 - \frac{2}{3}$ (d) $x^2 + 7$.

(10) Which one of the following polynomials has $(x+1)$ as a factor ?

(a) $x^3 - 4x^2 + x + 6$ (b) $x^3 + 4x^2 + x + 6$ (c) $x^3 - 4x^2 + x + 5$ (d) None of these.

(11) ———— is not a factor of the polynomial $x^3 - 3x^2 - x + 3$.

(a) $(x-3)$ (b) $(x-1)$ (c) $(x+1)$ (d) $(x+3)$

(12) If $f(x) = x^2 + 1$ and $g(x) = x^4 - 1$ then G.C.D of $f(x)$ and $g(x)$ is

(a) $x^2 - 1$ (c) $x - 1$
(b) $x^2 + 1$ (d) $x + 1$

(B) Descriptive Questions

(1) Find quotient and the remainder when $f(x)$ is divided by $g(x)$

(i) $f(x) = x^4 - 6x^3 + 7x^2 + 2x + 1$ and $g(x) = x + 3$.
(ii) $f(x) = x^5 + 2x^4 - 3x^2 + 4x + 2$ and $g(x) = x^2 + 2$.
(iii) $f(x) = x^4 + 6x^3 - 5x^2 + 4x - 1$ and $g(x) = x + 3$.
(iv) $f(x) = x^5 - 3x^4 - 2x^2 + 4x + 2$ and $g(x) = x^2 + 2x + 2$
(v) $f(x) = x^5 + 4x^4 + 2x^3 + 3x + 2$ and $g(x) = x^2 + x + 2$.
(vi) $f(x) = x^4 - 3x^3 + x^2 - 2x + 1$ and $g(x) = x^2 + x - 1$.
(vii) $f(x) = x^5 - 2x^4 + 3x^3 - 2x^2 + x + 1$ and $g(x) = x^2 + 2x + 1$.
(viii) $f(x) = x^4 - x^3 + 2x^2 + x - 6$ and $g(x) = x^2 + x + 1$.
(ix) $f(x) = x^5 + 3x^4 + x^2 + 2x + 2$ and $g(x) = x^2 + 2x + 1$.
(x) $f(x) = x^5 + 4x^4 - 3x^2 - 4x + 1$ and $g(x) = x^3 + x + 1$.

(2) Find G.C.D. of $f(x)$ and $g(x)$ in $R[x]$.

	$f(x)$	$g(x)$
(i)	$2x^3 - 13x^2 + 17x - 3$	$2x^3 + 5x^2 - 14x + 3$
(ii)	$x^4 - x^2 + x + 1$	$x^5 + x^4 - x^3 - x^2 + x + 1$
(iii)	$x^2 - 2x$	$x^4 - 4x^2$
(iv)	$x^8 - 1$	$x^6 - 1$
(v)	$x^4 - 7x^3 + 18x^2 - 20x + 8$	$x^2 - 3x + 2$
(vi)	$x^4 - 4x^3 + 3x^2 + 4x - 4$	$x^3 - 3x^2 - x + 3$
(vii)	$x^4 - 8x^3 + 22x^2 - 24x + 9$	$x^3 - 3x + 2$
$(viii)$	$x^4 - 5x^3 + 9x^2 - 7x + 2$	$x^3 - 2x^2 - x + 2.$
(ix)	$4x^3 - 9x^2 + 14x - 3$	$4x^4 - x^3 - 4x^2 + 5x + 1.$
$(x)f$	$x^8 - 1$	$x^{12} - 1.$
(xi)	$x^6 + x^3 - 2$	$x^6 - 1.$

xxxxxxxxxxxx

2.6 Practical 2.6: Polynomials (II)

2.6.1 Prerequisite of Practical 2.6

(1) Let $f(x) \in F[x]$ and $a \in F$. If $f(a) = 0$ then a is called as a **zero of** $f(x)$ (or a **root of** $f(x)$) if $f(a) = 0$.

(2) Let $f(x) \in F[x]$ and $a \in F$. From Factor Theorem, (prerequisite of practical 2.5 no. (17)), a is a zero of $f(x)$ if and only if $(x - a)$ is a factor of $f(x)$.

(3) Let $f(x) \in F[x]$ and $a \in F$. If $(x - a)^n$ divides $f(x)$ but $(x - a)^{n+1}$ does not divide $f(x)$ for some $n \in \mathbb{N}$, then n is called the **multiplicity** of the root a.

(4) A polynomial of degree n over F has at most n zeros, counting multiplicity.
(Let $f(x) \in F[x], f(x) \neq 0$. Proof is by induction on n. If $\deg(f(x)) = 0$, then $f(x)$ is a non-zero constant polynomial in the form $f(x) = c, c \neq 0$. Then $f(x)$ has no zero. Number of zeros is 0. If $\deg f(x)$ is 1, then $f(x) = a_0 + a_1 x$, where $a_1 \neq 0$. But then $\dfrac{-a_0}{a_1}$ is the only zero. Hence $f(x)$ has exactly one zero. Assume the theorem for polynomials of degree

$< n$. Now, consider $f(x) \in F[x]$ with $\deg f(x) = n, n \geq 2$.
If $f(x)$ has no zeros, then number of zeros $= 0 < n$ and hence the statement is true.
Suppose $f(x)$ has a zero, say a. Let k be the multiplicity of a. So $k \geq 1$ and we can write $f(x) = (x-a)^k q(x)$ where $(x-a)$ is not a factor of $q(x)$.
$f(x) = (x-a)^k q(x) \Longrightarrow \deg f(x) = \deg(x-a)^k + \deg q(x) = k + \deg q(x)$.
As, $\deg q(x) \geq 0, k \leq n$.
If $f(x)$ has no zeros other than a then we are done as we have, no. of zeros of $f(x) = k \leq n$.
On the other hand if $b \neq a$ and b is a zero of $f(x)$, then $0 = f(b) = (x-a)^k q(b)$.
Hence b is also a zero of $q(x)$ with the same multiplicity as it has for $f(x)$.
As $\deg q(x) < n$, by the Second Principle of Mathematical Induction, we know that $q(x)$ has at most $\deg q(x) = n-k$ zeros, counting multiplicity.
Hence $f(x)$ has at most $k + (n-k) = n$ zeros, counting multiplicity.)

(5) Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial in $\mathbb{R}[x]$. Let $\alpha \in \mathbb{C}$, be a root of $f(x)$, then conjugate of α is also a root of $f(x)$.

(6) Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial such that $a_0, a_1, \cdots, a_n \in \mathbb{Z}$.
If $\frac{a}{b} \in \mathbb{Q}$, is such that, $\gcd(a,b) = 1$ and $\frac{a}{b}$ is a root of $f(x)$ then $a \mid a_0$ and $b \mid a_n$.
(Since $\frac{a}{b}$ is a root of $f(x)$, we have, $a_0 + a_1\left(\frac{a}{b}\right) + +a_2\left(\frac{a}{b}\right)^2 + \cdots + a_n\left(\frac{a}{b}\right)^n = 0$.

$$\begin{aligned}
a_0 b^n + a_1 a b^{n-1} + a_2 a^2 b^{n-2} + \cdots + a_n a^n &= 0. \\
a_0 b^n + a_1 a b^{n-1} + a_2 a^2 b^{n-2} + \cdots + a_{n-1} a^{n-1} b &= -a_n a^n. \\
-a_0 b^n + a_1 a b^{n-1} - a_2 a^2 b^{n-2} + \cdots - a_{n-1} a^{n-1} b &= a_n a^n. \\
b\Big(- a_0 b^{n-1} - a_1 a b^{n-2} + a_2 a^2 b^{n-3} - \cdots - a_{n-1} a^{n-1}\Big) &= a_n a^n.
\end{aligned}$$

Hence $b \mid a_n a^n$. Since $\gcd(b,a) = 1, \gcd(b, a^n) = 1$. Hence $b \mid a_n a^n \Longrightarrow b \mid a_n$. Similarly,

$$a_0 b^n + a_1 a b^{n-1} + a_2 a^2 b^{n-2} + \cdots + a_n a^n = 0.$$

$$\begin{aligned}
a_0 b^n &= -a_1 a b^{n-1} - a_2 a^2 b^{n-2} - \cdots + a_{n-1} a^{n-1} b - a_n a^n. \\
a_0 b^n &= a\Big(- a_1 b^{n-1} - a_2 a b^{n-2} + \cdots - a_{n-1} a^{n-2} b - a_n a^{n-1}\Big).
\end{aligned}$$

Hence $a \mid a_0 b^n$. Since $\gcd(a,b) = 1, \gcd(a, b^n) = 1$. Hence $a \mid a_0 b^n \Longrightarrow a \mid a_0$.)

(7) Let $f(x) = a_0 + a_1 x + \cdots + x^n$ be a monic polynomial such that $a_0, a_1, \cdots, a_{n-1} \in \mathbb{Z}$. If $\frac{a}{b} \in \mathbb{Q}$ is a root of $f(x)$ then $b = \pm 1$.

(8) We can show that if p is a prime then $\sqrt{p}$ is an irrational number, using the polynomial $f(x) = x^2 - p$.

(9) **Relation between the zeros and coefficients of a polynomial** $f(x) \in F[x]$.

(i) Let $f(x) \in F[x]$ be a non-zero polynomial of degree n.. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n, a_n \neq 0$. And let $r_1, r_2, \cdots, r_n$ be the roots of $f(x)$.

sum of the roots taken one at a time	$= r_1 + r_2 + \cdots + r_n$	$= \dfrac{-a_{n-1}}{a_n}$
sum of the roots taken two at a time	$= r_1r_2 + r_1r_3 + \cdots + r_1r_n + r_2r_3 + \cdots$	$= \dfrac{a_{n-2}}{a_n}$
sum of the roots taken three at a time	$= r_1r_2r_3 + r_1r_2r_4 + \cdots$	$= \dfrac{-a_{n-3}}{a_n}$
and so on	....	....

(ii) If a polynomial is quadratic, $f(x) = ax^2 + bx + c$, and r_1 and r_2 are the roots of $ax^2 + bx + c$ then $r_1 + r_2 = \dfrac{-b}{a}$ and $r_1r_2 = \dfrac{c}{a}$.

(iii) If a polynomial is cubic, $f(x) = ax^3 + bx^2 + cx + d$, and r_1, r_2 and r_3 are the roots of $ax^3 + bx^2 + cx + d$ then $r_1 + r_2 + r_3 = \dfrac{-b}{a}$ and $r_1r_2 + r_1r_3 + r_2r_3 = \dfrac{c}{a}, r_1r_2r_3 = \dfrac{-d}{a}$.

(iv) The quadratic polynomial with roots r_1, r_2 is given by $f(x) = x^2 - (r_1 + r_2)x + r_1r_2$.

(v) The cubic polynomial with roots r_1, r_2, r_3 is given by $f(x) = x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3$.

(10) **Irreducible Polynomial**: A non-constant polynomial $f(x) \in F[x]$ is said to be **irreducible** over F if, whenever $f(x)$ is expressed as a product $f(x) = g(x) \cdot h(x)$, with $g(x), h(x) \in F[x]$, then $g(x)$ or $h(x)$ is a constant polynomial in $F[x]$.
Note:

(i) A nonconstant polynomial $f(x) \in F[x]$ is said to be irreducible if $g(x) \in F[x]$ and $g(x)|f(x)$ then either $g(x) = k$ or $g(x) = kf(x)$ for some nonzero $k \in F$.

(ii) A nonconstant polynomial $f(x) \in F[x]$ is said to be irreducible if $f(x)$ can not be expressed as a product of two polynomials of lower degree.

A nonzero, nonconstant polynomial of $F[x]$ that is not irreducible over F is called **reducible** over F.

(i) The polynomial $f(x) = 2x^2 + 4 \in \mathbb{Q}[x]$. $f(x)$ is irreducible over $\mathbb{Q}$.

(ii) The polynomial $f(x) = 2x^2 + 4 \in \mathbb{R}[x]$. $f(x)$ is irreducible over $\mathbb{R}$.

(iii) The polynomial $f(x) = 2x^2 + 4 \in \mathbb{C}[x]$. $f(x) = 2(x + \sqrt{2}\ i)(x - \sqrt{2}\ i)$ is reducible over $\mathbb{C}$.

(iv) The polynomial $f(x) = x^2 - 2 \in \mathbb{Q}[x]$ is irreducible over $\mathbb{Q}$.

(v) The polynomial $f(x) = x^2 - 2 \in \mathbb{R}[x]$. We can write $f(x) = (x - \sqrt{2})(x + \sqrt{2})$ and hence $f(x)$ is reducible over $\mathbb{R}$.

(11) Let $p(x), a(x) \in F[x]$. If $p(x)$ is irreducible over F and $p(x) \not| a(x)$, then $\gcd(p(x), a(x)) = 1$.
(Suppose $\gcd(p(x), a(x)) = d(x)$. Therefore, $d(x)|p(x)$ and $d(x)|a(x)$.

$d(x)|p(x) \Longrightarrow]$ $d(x) = k$ or $d(x) = kp(x)$ for some nonzero $k \in F$. (from the definition of irreducible polynomial no (i)).

If $d(x) = kp(x)$, then $d(x)|a(x) \Longrightarrow kp(x)|a(x) \Longrightarrow p(x)|a(x)$. This is a contradiction.

Hence $d(x) = k$ for some nonzero $k \in F$.
Therefore, $\gcd(p(x), a(x)) = k$.

Hence $\gcd(p(x), a(x)) = 1$. (as $1 \mid k$ and $1 = k * \frac{1}{k} \Longrightarrow k \mid 1$)

(12) Let $p(x), a(x), b(x) \in F[x]$. If $p(x)$ is irreducible over F and $p(x) \mid a(x)b(x)$, then $p(x) \mid a(x)$ or $p(x) \mid b(x)$.
(If $p(x) \mid a(x)$ then we are through. Suppose $p(x) \not| a(x)$. Then $p(x)$ irreducible $\Longrightarrow$ $\gcd(p(x), a(x)) = 1$.
Therefore, there exist $m(x), n(x) \in F[x]$ such that $p(x)m(x) + a(x)n(x) = 1$. Multiplying both sides by $b(x)$, we get, $p(x)m(x)b(x) + a(x)n(x)b(x) = b(x)$.
Since $p(x) \mid a(x)b(x)$ and $p(x) \mid p(x)$, we have, $p(x) \mid (p(x)m(x)b(x) + a(x)n(x)b(x))$. This implies $p(x) \mid b(x)$.

(13) **Unique Factorization Theorem**: Let $f(x) \in F[x]$, be such that $\deg f(x) \geq 1$. Then $f(x)$ can be expressed as k times, a product of irreducible monic polynomials in $F[x]$ for some $k \in F$. Further, this representation is unique except for the order in which the factors occur.
(Proof is by induction on $\deg f(x)$.
Let the statement be: Every polynomial in $F[x]$ with degree $n, n \geq 1$, can be expressed as k times, a product of irreducible monic polynomials in $F[x]$ for some $k \in F$. $(*)$
If $n = 1, f(x) = a_0 + a_1x, a_1 \neq 0$. Then we can write $f(x) = a_1 \left(\dfrac{a_0}{a_1} + x\right)$.
Here, we have expressed $f(x)$ as k times, a product of irreducible monic polynomials where $k = a_1$ and one monic, irreducible polynomial is $\dfrac{a_0}{a_1} + x$.
We assume the statement for all polynomials with degree $< n$.
Let $f(x) \in F[x]$ be such that $\deg f(x) = n, f(x) = a_0 + a_1x + \cdots + a_nx^n, a_n \neq 0$.

If $f(x)$ is irreducible, then we will write $f(x) = a_nf_1(x)$, where $f_1(x)$ is monic and irreducible.

If $f(x)$ is not irreducible, then we can write $f_1(x) = g(x)h(x)$, where $g(x), h(x) \in F[x]$ and $\deg g(x) < \deg f(x), \deg h(x) < \deg f(x)$ (from the definition of irreducible polynomial no. (ii)).
So, $1 \leq \deg g(x) < \deg f(x)$ and $1 \leq \deg h(x) < \deg f(x)$.
By Induction, $g(x)$ can be expressed as k_1 times, a product of monic irreducible polynomials for some $k_1 \in F$ and $h(x)$ also can be expressed as k_2 times a product of monic irreducible polynomials for some $k_2 \in F$.
Clearly, then, $f(x)$ is expressed as $k_1 \cdot k_2$ times a product of monic irreducible polynomials.

Therefore the statement $(*)$ is true for all $n \in \mathbb{N}$.
Now, we will prove that this representation is unique except for the order in which the factors occur.
Suppose $f(x) = \alpha p_1(x)p_2(x)\cdots p_r(x) = \beta q_1(x)q_2(x)\cdots q_s(x)$ where $\alpha, \beta \in F$ and $p_!(x), p_2(x), \cdots, p_r(x$
$F[x]$ are monic irreducible polynomials in $F[x]$.
Since $p_1(x), \cdots, p_r(x)$ are monic, the leading coefficient of L.H.S. is α.
Since $q_1(x), \cdots, q_s(x)$ are monic, the leading coefficient of R.H.S. is β.
Therefore $\alpha = \beta$.
Hence $p_1(x)p_2(x)\cdots p_r(x) = q_1(x)q_2(x)\cdots q_s(x)$. $(*)$
Therefore $p_1(x) \mid q_1(x)q_2(x)\cdots q_s(x)$.
Since $p_1(x)$ is irreducible, $p_1(x) \mid q_1(x)q_2(x)\cdots q_s(x) \Longrightarrow p_1(x) \mid q_1(x)$ or $p_1(x) \mid q_2(x)\cdots$ or $p_1(x) \mid q_s(x)$.(by statement no. (12) in the above prerequisite)

That is, $p_1(x) \mid q_i(x)$ for some $i, 1 \leq i \leq s$.
By renumbering, we may assume $i = 1$, that is, $p_1(x) \mid q_1(x)$.
Now, $q_1(x)$ is irreducible implies $p_1(x) = k$ or $p_1(x) = kq_1(x)$ for some $k \in F$. (from definition of irreducible polynomial no. (i))
As $p_1(x)$ is irreducible, $p_1(x) \neq k$. (irreducible polynomials are non-constants)
Therefore $p_1(x) = kq_1(x)$ for some $k \in F$.
But $p_1(x)$ and $q_1(x)$ are monic. This implies $k = 1$.
Hence $p_1(x) = q_1(x)$.
Thus, $p_1(x)p_2(x)\cdots p_r(x) = p_1(x)q_2(x)\cdots q_s(x)$.
By Cancellation law, (by prerequisite of practical 2.5, statement no. (12)) $p_2(x)\cdots p_r(x) = q_2(x)\cdots q_s(x)$.
Now, we repeat the argument above with $p_2(x)$ in place of $p_1(x)$.
If $r < s$, then after r such steps we will have 1 on the left and a non-constant polynomial on the right.
This is a contradiction.
If $r > s$, then after r such steps we will have 1 on the right and a non-constant polynomial on the left.
This is a contradiction.
So, $r = s$ and $p_i(x) = q_i(x)$ for all $i, 1 \leq i \leq r$ after suitable renumbering of $q(x)'s$.
Thus we have proved the required uniqueness.)

2.6.2 PRACTICAL 2.6

(A) Objective Questions

Choose correct alternative in each of the following:

(1) A quadratic polynomial whose roots are -5 and 7 is

(a) $x^2 - 2x - 35$ (c) $x^2 + 2x + 35$
(b) $x^2 - 2x + 35$ (d) $x^2 + 2x - 35$

(2) A polynomial whose roots are $-2, 3$ and 7 is

(a) $x^3 - 8x^2 + x + 42$ (c) $x^3 + 8x^2 + x + 42$
(b) $x^3 - 8x^2 + x - 42$ (d) $x^3 - 8x^2 - x - 42$

(3) If one of the roots of the quadratic polynomial $(k-1)x^2 + kx + 2$ is 5, then the value of k is

(a) $\frac{-23}{30}$ (b) $\frac{23}{30}$ (c) $\frac{-23}{25}$ (d) -5

(4) For a cubic polynomial $ax^3 + bx^2 + cx + d$ with roots r_1, r_2, r_3 we have

(a) $r_1r_2 + r_2r_3 + r_1r_3 = \frac{c}{a}$

(b) $r_1 + r_2 + r_3 = \frac{b}{a}$

(c) $r_1r_2r_3 = \frac{d}{a}$

(d) $r_1r_2 + r_2r_3 + r_1r_3 = -\frac{c}{a}$

(5) If r_1, r_2, r_3 are roots of polynomial $7x^3 + 21x^2 - 4x + 6$ then ....

(a) $r_1r_2 + r_2r_3 + r_1r_3 = \frac{4}{7}$

(b) $r_1r_2r_3 = 6$

(c) $r_1 + r_2 + r_3 = 3$

(d) $r_1 + r_2 + r_3 = -3$

(6) If r_1, r_2, r_3 are roots of polynomial $8x^3 - 18x^2 + 3x - 4$ then ....

(a) $r_1r_2r_3 = -\frac{3}{8}$

(b) $r_1r_2r_3 = -\frac{1}{8}$

(c) $r_1r_2r_3 = \frac{18}{8}$

(d) $r_1r_2r_3 = \frac{1}{2}$

(7) Which is the root of the polynomial $6x^3 - 49x^2 + 51x - 14$.

(a) $-\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $-\frac{2}{3}$ (d) -7.

(8) The sum of the roots of the polynomial whose roots are twice the roots of the polynomial $x^2 + 17x + 11$ is

(a) 17 (b) 11 (c) 34. (d) -34

(9) If $-3+4i$ is a root of the polynomial $f(x)$ of degree 2 then ———— is also a root of $f(x)$.

(a) $-3-4i$ (b) $3-4i$ (c) $3+4i$ (d) 3

(10) The sum of all n, n^{th} roots of unity, for $n \in \mathbb{N}$ ....

(a) n (b) $-n$ (c) 0 (d) $n-1$

(11) The roots of the polynomial $x^3 - 5x^2 - 16x + 80$ are

(a) $-4, 4, -5$ (b) $4, 4, 5$ (c) $-4, 4, 5$ (d) $4, 4, -5$

(12) ———— is not a root of polynomial $x^3 - 3x^2 - x + 3$.

(a) -3 (b) 3 (c) 1 (d) -1

(B) Descriptive questions

(1) Find the multiplicity of each root of polynomial $f(x)$

(i) x^4-6x^2-8x-3
(ii) $x^4-5x^3+9x^2-7x+2$.
(iii) $x^4+2x^3-3x^2-4x+4$.
(iv) $x^4-3x^3+x^2+3x-2$.
(v) $x^4-7x^3+18x^2-20x+8$
(vi) $x^4-12x^3+34x^2+12x-35$
(vii) $x^4-12x^3+46x^2-60x+25$
(viii) $x^4-14x^3+61x^2-84x+36$
(ix) $x^4-11x^3+29x^2+11x-30$
(x) $x^4-4x^3+3x^2+4x-4$
(xi) $x^4-x^3-3x^2+5x-2$

(2) Find all roots of $f(x)$ if sum of its two roots is zero.

(i) x^3-2x^2-4x+8
(ii) $x^3-2x^2-16x+32$
(iii) $x^3-2x^2-25x+50$
(iv) $x^3-5x^2-4x+20$
(v) $x^3-3x^2-4x+12$
(vi) $x^3-2x^2-9x+18$
(vii) x^3-2x^2-4x+8
(viii) x^3-2x^2-x+2

(3) If r_1, r_2 and r_3 are roots of $f(x)$ find k, if $r_1+r_2=r_3$ and factorize $f(x)$.

(i) x^3-4x^2-4x+k
(ii) $x^3+12x^2+44x+k$
(iii) $x^3+6x^2+11x+k$
(iv) $x^3-14x^2+55x+k$
(v) $x3-10x2+29x+k$
(vi) x^3-2x^2-5x+k
(vii) x^3-4x^2+x+k
(viii) x^3+2x^2-x-k
(ix) x^3-6x^2+5x+k
(x) x^3+2x^2-5x-k

(4) In each of the following examples r_1, r_2 are two roots of the given polynomial $f(x)$ satisfying certain condition. Find k and factorize $f(x)$.

(i) $f(x)=x^3-x^2-4x+k$, such that $r_1+r_2=0$.
(ii) $f(x)=2x^3+3x^2+kx-1$, such that $r_1+r_2=2$.
(iii) $f(x)=x^3-7x+k$, such that $r_1+r_2=-1$.
(iv) $f(x)=x^3-2x^2-16x+k$ such that $r1+r2=0$.

(5) Find all roots of $f(x)$ if sum of its two roots is the third root.

(i) $x^3-8x^2+20x-16$
(ii) x^3+6x^2-x-30
(iii) $x^3-4x^2-20x+48$
(iv) $x^3-10x^2+31x-30$
(v) $x^3-12x^2+44x-48$
(vi) $x^3+4x^2-4x+16$
(vii) $x^3+12x^2+45x+54$

(6) Determine whether following polynomial $f(x)$ has a rational root.

(i) $x^4+x^3+x^2+x+1$
(ii) x^4-5x^2+4
(iii) $5x^3-15x^2+7x-21$
(iv) x^3-4x+2

(7) If r_1, r_2 and r_3 are roots of polynomial $f(x)=x^3-3x^2+4x-2$, without actually calculating the values of r_1, r_2 and r_3 write the polynomial with roots $4r_1, 4r_2$ and $4r_3$.

(8) If r_1, r_2 and r_3 are roots of polynomial $f(x) = x^3+7x^2+4x-3$, without actually calculating the values of r_1, r_2 and r_3 write the polynomial with roots $5r_1, 5r_2$ and $5r_3$.

(9) If r_1, r_2 and r_3 are roots of polynomial $f(x) = x^3+x^2+2x-1$, without actually calculating the values of r_1, r_2 and r_3 write the polynomial with roots $2r_1, 2r_2$ and $2r_3$.

xxxxxxxxxxxx

2.7 Practical 1.7: Miscellaneous theory questions

2.7.1 Miscellaneous theory questions from unit I

Unit I

(1) Prove that '1' is the least element of $\mathbb{N}$.

(2) Statements of well-ordering property of non-negative integers.

(3) State and prove the division algorithm.

Or

Given integers a, b $(b > 0)$, prove that there exist unique integers q, r satisfying,

$$a = bq + r,\ 0 \leq r < b.$$

(4) Define divisibility in integers. Define GCD and LCM of two nonnegative integers. Define relatively prime integers.

(5) Prove that every non-zero integer which is $\neq \pm 1$ has at least one prime divisor.

(6) Prove that if $a, b \in \mathbb{Z}$ and atleast one of a, b is non-zero, then there exists $x, y \in \mathbb{Z}$ such that $(a, b) = xa + yb$.

(7) Prove that integers a, b are relatively prime if and only if then there exists $x, y \in \mathbb{Z}$ such that $1 = xa + yb$.

(8) State and prove Euclid's Lemma.

Or

Prove that if integers a, b, c, are such that $a|bc$ and $(a, b) = 1$, then $a|c$.

(9) Prove that if $a, b \in \mathbb{Z}$ and $b \neq 0$ then $(a, b) = (b, a + xb)\ \forall\ x \in \mathbb{Z}$.

(10) State Euclid's algorithm for finding GCD of two non-negative integers.

(11) What do you mean by a prime numbers and a composite number?

(12) Prove that if $a, b \in \mathbb{Z}$ and prime p divides product ab, then either $p|a$ or $p|b$.

(13) Prove that $[a, b] \times (a, b) = ab \; \forall \; a, b \in \mathbb{N}$.

(14) State and prove The Fundamental Theorem of Arithmetic (Unique factorization theorem for integers)

Or

Prove that every positive integer $n > 1$ can be expressed as a product of prime numbers and this factorization is unique apart from the order of prime factors.

(15) Prove that the set of primes is infinite.

(16) Prove that there are infinitely many primes of the form $4n - 1$ or $4n + 1$ or $6n - 1$.

(17) Prove that if prime p does not divide integer a then $(a, p) = 1$.

(18) What do you mean by a congruence relation?

(19) Show that any two integers a, b are congruent modulo n if and only if they leave the same remainder when divided by n.

(20) Let $a, b, c, d \in \mathbb{Z}$ and n be a positive integer. Prove that

(i) $a \equiv a \pmod{n}$

(ii) If $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$.

(iii) If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$.

(iv) If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then $a + c \equiv b + c \pmod{n}$, $ac \equiv bc \pmod{n}$, $a + c \equiv b + d \pmod{n}$, and $ac \equiv bd \pmod{n}$.

(v) If $a \equiv b \pmod{n}$ then $a^k \equiv b^k \pmod{n} \; \forall \; k \in \mathbb{N}$.

(21) Prove that if p is prime then $\phi(p^k) = p^k - p^{k-1} = p^k \left(1 - \frac{1}{p}\right) \; \forall \; k \in \mathbb{N}$.

(22) Prove that if n is a positive integer with a prime factorization $n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ then $\phi(n) = \left(p_1^{k_1} - p_1^{k_1-1}\right)\left(p_2^{k_2} - p_2^{k_2-1}\right) \cdots \left(p_m^{k_m} - p_m^{k_m-1}\right) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right)$.

(23) Prove that if the GCD of integers m, n is 1 then $\phi(mn) = \pi(m)\phi(n)$.

(24) State Euler's theorem, Fermat's theorem and Wilson's theorem.

2.7.2 Miscellaneous theory questions from unit II

(1) What do you mean by a function from $\mathbb{R}$ to RR, Define domain, co-domain and range of a function. Define composite of two functions.

(2) Define injective, surjective and bijective functions, invertible function.

(3) Prove that if $f : A \to B, g : B \to C$ and $h : C \to D$, then $h \circ (g \circ f) = (h \circ g) \circ f$.

(4) Let $f : A \to B$ and S, T be subsets of A. Prove that

(i) If $S \subseteq T$ then $f(S) \subseteq f(T)$.
(ii) $f(S \cup T) = f(S) \cup f(T)$.
(iii) $f(S \cap T) \subseteq f(S) \cap f(T)$.
(iv) $S \subseteq f^{-1}(f(S))$.

(5) Let $f : A \to B$ and X, Y be subsets of B. Prove that

(i) If $X \subseteq Y$ then $f^{-1}(X) \subseteq f^{-1}(Y)$.
(ii) $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$.
(iii) $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$.
(iv) $f\left(f^{-1}(X)\right) \subseteq X$.

(6) Let $f : A \to B$ and $g : B \to C$. Prove that

(i) If f, g are injective then so is $g \circ f$.
(ii) If f, g are surgective then so is $g \circ f$.
(iii) If f, g are bijective then so is $g \circ f$.
(iv) If $g \circ f$ is injective then so is f.
(v) If $g \circ f$ is surjective then so is g.

(7) Prove that if $f : A \to B$ is invertible then f has unique inverse.

(8) Prove that if $f : A \to B$ is invertible if and only if f is bijective.

(9) Define an equivalence relation, equivalence classes, partition of a set.

(10) Suppose R is an equivalence relation in non-empty set X. Prove that distinct equivalence classes of R form a partition of set X.

(11) Suppose $P = \{X_i\}_{i \in \Lambda}$ is a partition of a non-empty set X. Prove that partition P induces an equivalence relation R in set X for which the equivalence classes are precisely $\{X_i\}_{i \in \Lambda}$.

(12) Prove that for each $n \in \mathbb{N}$, the congruence relation modulo n is an equivalence relation in $\mathbb{Z}$.

2.7.3 Miscellaneous theory questions from unit III

(1) Define polynomial over F where $F = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, degree of a polynomial, leading coefficient, monic polynomial.

(2) Let $f(x), g(x)$ and $h(x)$ be a polynomial over $\mathbb{R}$. Prove that for each $x \in \mathbb{R}$,

(i) $(f(x) + g(x)) + h(x) = f(x) + (g(x) + h(x))$
(ii) $f(x) + g(x) = g(x) + f(x)$
(iii) $f(x) + 0 = 0 + f(x) = f(x)$
(iv) $(f(x) \cdot g(x)) \cdot h(x) = f(x) \cdot (g(x) \cdot h(x))$
(v) $f(x) \cdot g(x) = g(x) \cdot f(x)$

(vi) $f(x) \cdot 1 = 1 \cdot f(x) = f(x)$

(3) Prove that given if non-zero polynomials $f(x), g(x) \in F[x]$ then $\deg(f(x) \cdot g(x)) = \deg(f(x)) + \deg(g(x))$.

(4) Define divisibility in polynomials.

(5) State Division algorithm in $F[x]$

(6) Let $f(x), g(x)$ and $h(x)$ be a polynomial over F and $f(x) \neq 0$. Prove that for each $x \in \mathbb{R}$,

(i) $f(x)|g(x) \implies f(x)|g(x) \cdot h(x)$

(ii) $f(x)|g(x) \implies cf(x)|g(x)$ for each non-zero $c \in F$

(iii) $f(x)|g(x)$ and $g(x)|h(x)$ implies $f(x)|h(x)$

(iv) $f(x)|g(x)$ and $f(x)|h(x)$ implies $f(x)|\lambda(x)g(x) + \mu(x)h(x)$ for all $\lambda(x), \mu(x) \in F[x]$

(v) If $f(x)|g(x)$ and $g(x) \neq 0$ then $\deg(f(x)) \leq \deg(g(x))$.

(7) Define GCD of two polynomials $F[x]$.

(8) State and prove the remainder theorem for polynomials in $F[x]$.

Or

Let $a \in F$ and $f(x) \in F[x]$. Prove that the remainder when $f(x)$ is divided by $x - a$ in $F[x]$ is $f(a)$.

(9) State and prove the factor theorem for polynomials in $F[x]$.

Or

Let $a \in F$ and $f(x) \in F[x]$. Prove that $x - a$ is a factor of $f(x) \in F[x]$ if and only if $f(a) = 0$.

(10) Define root of a polynomial over F.

(11) Prove that a polynomial of degree n over field F has atmost n roots in F.

(12) State and prove the rational root theorem for polynomials having integer coefficients.

Or

Prove that if rational number $\frac{a}{b}, \gcd(a, b) = 1$ is a root of polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, a_0, a_1, a_n \in Z, a_n \neq 0$ then $a|a_0$ and $b|a_n$.

(13) Prove that a rational root of a monic polynomial with integer coefficients is an integer.

(14) Let $p(x), a(x) \in F[x]$ where $F = \mathbb{Q}$ or $\mathbb{R}$ or $\mathbb{C}$. If $p(x)$ is irreducible over F and $p(x) \not| a(x)$, then $\gcd(p(x), a(x)) = 1$.

(15) Prove that if complex number z ia s root of polynomial $f(x) \in \mathbb{R}[x]$ then its complex conjugate $\overline{z}$ is also a root of $f(x)$.

(16) Prove that a non-constant polynomial $f(x) \in \mathbb{R}[x]$ can be expressed as a product of Linear and quadratic polynomials (which can't be factorized further).

Chapter 3

(USMT/UAMT 201) Calculus II

3.1 Practical 1.1: Limits and Continuity

3.1.1 Prerequisite of Practical 1.1:

(1) **Limit Point**: Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$. Then c is called a limit point of D if every neighbourhood of c contains at least one point of D other than c.

(2) Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $f : D \longrightarrow \mathbb{R}$ be a function. We say that a **limit** of f as x tends to c exists if there is a real number L satisfying the following $\epsilon - \delta$ condition: For every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in D$ and $0 < |x - c| < \delta \implies |f(x) - L| < \varepsilon$. We then write $\lim\limits_{x \longrightarrow p} f(x) = L$.

(3) For example, using the definition of limit, let us show that $\lim\limits_{x \to 3} 14 - 2x = 8$.
We show that, $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that $0 < |x - 3| < \delta \Longrightarrow |(14 - 2x) - 8| < \varepsilon$.
Let $\varepsilon > 0$ be given.
We want to find $\delta > 0$ such that $0 < |x - 3| < \delta \Longrightarrow |(14 - 2x) - 8| < \varepsilon$.
Suppose $0 < |x - 3| < \delta$, for some $\delta > 0$.

$$\text{Consider } |(14 - 2x) - 8| = |6 - 2x| = 2|3 - x| = 2|x - 3| < 2\delta.$$

Hence, if we choose $\delta > 0$ such that $2\delta < \epsilon$ then we have,
$|(14 - 2x) - 8| < 2\delta < \epsilon$.
Hence, if $0 < \delta < \dfrac{\epsilon}{2}$ then $0 < |x - 3| < \delta \Longrightarrow |(14 - 2x) - 8| < \epsilon$.
Thus, $\exists\, \delta > 0$ such that $0 < |x - 3| < \delta \Longrightarrow |(14 - 2x) - 8| < \epsilon$.
Hence, $\lim\limits_{x \to 3} 14 - 2x = 8$.

(4) Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. If $f : D \longrightarrow \mathbb{R}$ and $\lim\limits_{x \to c} f(x)$ exists then it is unique.

(5) Let $D \subseteq \mathbb{R}, c \in \mathbb{R}$ be a limit point of D. and let $f : D \longrightarrow \mathbb{R}$ be such that $\lim\limits_{x \to c} f(x) = L$ for some $L \in \mathbb{R}$. Then

(i) If $L > 0$, then there is $\delta > 0$ such that $f(x) > 0$ for all $x \in D$ satisfying $0 < |x-c| < \delta$.

(ii) If $L < 0$, then there is $\delta > 0$ such that $f(x) < 0$ for all $x \in D$ satisfying $0 < |x-c| < \delta$.

(iii) Hence if $L \neq 0$ then there is $\delta > 0$ such that $f(x) \neq 0$ for all $x \in D$ satisfying $0 < |x-c| < \delta$.

(6) Let $D \subseteq \mathbb{R}, c \in \mathbb{R}$ be a limit point of D. and let $f : D \longrightarrow \mathbb{R}$ be such that $\lim\limits_{x \to c} f(x) = L$ for some $L \in \mathbb{R}$. Then there exist $K, \delta > 0$ such that

$$|f(x)| \leq K \text{ for all } x \in D \text{ satisfying } 0 < |x-c| < \delta.$$

(7) Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. If $f : D \longrightarrow \mathbb{R}$ and $\lim\limits_{x \to c} f(x) = L$ for some $L \in \mathbb{R}$. If there is $\delta_1 > 0$, such that

(i) $f(x) > 0$ for all $x \in D$ satisfying $0 < |x-c| < \delta_1$ then $L \geq 0$.

(ii) $f(x) < 0$ for all $x \in D$ satisfying $0 < |x-c| < \delta_1$ then $L \leq 0$.

(8) Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. If $f, g : D \longrightarrow \mathbb{R}$ and $L, M \in \mathbb{R}$. Then

(i) $\lim\limits_{x \to c} f(x) = L$ if and only if $\lim\limits_{h \to 0} f(c+h) = L$.

(ii) if $\lim\limits_{x \to c} f(x) = 0$ and there exists $K, \delta_1 > 0$ such that $|g(x)| \leq K$ for all $x \in D$ satisfying $0 < |x-c| < \delta_1$ then $\lim\limits_{x \to c} f(x)g(x) = 0$.

(iii) if $f(x) \leq g(x)$ for all $x \in \mathbb{D}$ and $\lim\limits_{x \longrightarrow p} f(x) = L$, $\lim\limits_{x \longrightarrow p} g(x) = M$ then $L \leq M$.

(iv) If $\lim\limits_{x \to p} f(x) = L$ then $\lim\limits_{x \to p} |f(x)| = |L|$. (Converse not true)

(9) **Algebra of Limits:** Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $L, M \in \mathbb{R}$ and $f, g : D \longrightarrow \mathbb{R}$ be functions such that $\lim\limits_{x \to c} f(x) = L$ and $\lim\limits_{x \to c} g(x) = M$. Then

(1) Sum Rule: $\lim\limits_{x \to c} (f+g)(x) = L + M$.

(2) Constant Multiple: $\lim\limits_{x \to c} (rf)(x) = rL$ for every $r \in \mathbb{R}$.

(3) Product Rule: $\lim\limits_{x \to c} (fg)(x) = L\ M$.

(4) Quotient Rule: $\lim\limits_{x \to c} \left(\dfrac{f}{g}\right)(x) = \dfrac{L}{M}$, whenever $g(x) \neq 0, \forall\ x \in D$ and $M \neq 0$.

(10) **Sandwich Theorem for limit of a function**
Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. Also, let $L \in \mathbb{R}$ and let $f, g, h : D \longrightarrow \mathbb{R}$ be such that $f(x) \leq h(x) \leq g(x)$, for all $x \in D$ and $\lim\limits_{x \to c} f(x) = L = \lim\limits_{x \to c} g(x)$. Then

$$\lim_{x \to c} h(x) = L.$$

(11) **Left-hand Limit:** Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of $D \cap (-\infty, c)$, that is, every neighbourhood of c contains at least one point say x from D such that $x < c$. Also, let $f : D \longrightarrow \mathbb{R}$ be a function. We say that a **left (hand) limit** of f as x tends to c (from the left) exists if there is $L \in \mathbb{R}$ satisfying the following $\varepsilon - \delta$ condition: For every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in D, c - \delta < x < c \Longrightarrow |f(x) - L| < \varepsilon$. We then write

$$f(x) \longrightarrow L \text{ as } x \longrightarrow c^- \quad \text{or} \quad \lim_{x \to c^-} f(x) = L.$$

if $\forall\ \varepsilon > 0,\ \exists\ \delta > 0$ such that $p - \delta < x < p \Longrightarrow |f(x) - L| < \varepsilon$.

(12) **Right-hand Limit:** Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of $D \cap (c, \infty)$, that is, every neighbourhood of c contains at least one point say x from D such that $c < x$. Also, let $f : D \longrightarrow \mathbb{R}$ be a function. We say that a **right (hand) limit** of f as x tends to c (from the right) exists if there is $L \in \mathbb{R}$ satisfying the following $\varepsilon - \delta$ condition: For every $\varepsilon > 0$ there exists $\delta > 0$ such that $x \in D, c < x < c + \delta \Longrightarrow |f(x) - L| < \varepsilon$. We then write

$$f(x) \longrightarrow L \text{ as } x \longrightarrow c^+ \quad \text{or} \quad \lim_{x \to c^+} f(x) = L$$

(13) Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be such that c is a limit of $D \cap (-\infty, c)$ as well as of $D \cap (c, \infty)$. Also, let $f : D \longrightarrow \mathbb{R}$ be a function. Then
$\lim\limits_{x \longrightarrow c} f(x)$ exists $\Longleftrightarrow$and i $\lim\limits_{x \to c^-} f(x)$ and $\lim\limits_{x \to c^+} f(x)$ exist and are equal.
In this case, $\lim\limits_{x \longrightarrow c} f(x) = \lim\limits_{x \to c^-} f(x) = \lim\limits_{x \to c^+} f(x)$.

(14) The following limits do not exist.

(i) $\lim\limits_{x\longrightarrow 0}\dfrac{1}{x}$ (ii) $\lim\limits_{x\longrightarrow 0}\dfrac{1}{x^2}$ (iii) $\lim\limits_{x\longrightarrow 1}\dfrac{1}{x-1}$ (iv) $\lim\limits_{x\longrightarrow 0}\dfrac{1}{\sqrt{x}}$

(15) Let $D\subseteq\mathbb{R}$ and let $c\in\mathbb{R}$ be a limit point of D. Also, let $f:D\longrightarrow\mathbb{R}$ be a function. We say that $f(x)$ tends to infinity as x tends c, if the following $\alpha-\delta$ **condition** holds: for every $\alpha>0$, there is $\delta>0$ such that $x\in D$ and $0<|x-c|<\delta\Longrightarrow f(x)>\alpha$. We then write

$$\lim_{x\to c} f(x)=\infty.$$

(16) Let $D\subseteq\mathbb{R}$ and let $c\in\mathbb{R}$ be a limit point of D. Also, let $f:D\longrightarrow\mathbb{R}$ be a function. We say that $f(x)$ tends to -infinity as x tends c, if the following $\beta-\delta$ **condition** holds: for every $\beta<0$, there is $\delta>0$ such that $x\in D$ and $0<|x-c|<\delta\Longrightarrow f(x)<\beta$. We then write

$$\lim_{x\to c} f(x)=-\infty.$$

(17) Suppose $D\subseteq\mathbb{R}$ is not bounded above and $f:D\longrightarrow\mathbb{R}$ is a function. We say that a **limit** of f as x tends to ∞ exists if there is $L\in\mathbb{R}$ satisfying the following $\epsilon-\alpha$ condition: for every $\varepsilon>0$, there is $\alpha>0$ such that $x\in D, x>\alpha\implies|f(x)-L|<\varepsilon$. We then write

$$f(x)\longrightarrow L \text{ as } x\longrightarrow\infty \quad\text{or}\quad \lim_{x\longrightarrow\infty} f(x)=L.$$

(18) Suppose $D\subseteq\mathbb{R}$ is not bounded below and $f:D\longrightarrow\mathbb{R}$ is a function. We say that a **limit** of f as x tends to $-\infty$ exists if there is $L\in\mathbb{R}$ satisfying the following $\epsilon-\beta$ condition: for every $\varepsilon>0$, there is $\beta<0$ such that $x\in D, x<\beta\implies|f(x)-L|<\varepsilon$. We then write

$$f(x)\longrightarrow L \text{ as } x\longrightarrow-\infty \quad\text{or}\quad \lim_{x\longrightarrow-\infty} f(x)=L.$$

(19) Some limits:

(i) $\lim\limits_{x\longrightarrow\infty}\dfrac{1}{x}=0=\lim\limits_{x\longrightarrow-\infty}\dfrac{1}{x}$.

(ii) $\lim\limits_{x\longrightarrow\infty}x^3=\infty$.

(iii) $\lim\limits_{x\longrightarrow-\infty}x^3=-\infty$.

(iv) $\lim\limits_{x\longrightarrow 0^+}\dfrac{1}{x}=\infty$.

(v) $\lim\limits_{x\longrightarrow 0^-}\dfrac{1}{x}=-\infty$.

(20) Let $D\subseteq\mathbb{R}$ and let $c\in D$. Also let $f:D\longrightarrow\mathbb{R}$ be a function. We say that f is **continuous** at c if f satisfies the following $\varepsilon-\delta$ condition:
For every $\varepsilon>0$, $\exists\,\delta>0$ such that $x\in D, |x-c|<\delta\implies|f(x)-f(c)|<\varepsilon$.
In case f is continuous at every $c\in D$, we say that f is continuous on D.

(21) Let $D \subseteq \mathbb{R}$ and $c \in D$. A function $f : D \to \mathbb{R}$ is said to be **discontinuous** at c if f is not continuous at c, that is, if there exists $\varepsilon > 0$ such that for all $\delta > 0$, $\exists\ x \in D$ with $|x - c| < \delta$ but $|f(x) - f(c)| \geq \varepsilon$.

(22) Some continuous functions and discontinuous functions.

(i) Let $a, b \in \mathbb{R}$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = ax + b$ for all $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$.

(ii) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = |x|$ for all $x \in \mathbb{R}$. Then f is continuous on $\mathbb{R}$.

(iii) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = [x]$ for all $x \in \mathbb{R}$. Then f is continuous at every $c \in \mathbb{R} \setminus \mathbb{Z}$.

(iv) $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ -x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Then f is continuous only at $c = 0$.

(v) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = [x]$ for all $x \in \mathbb{R}$. Then f is discontinuous at every $c \in \mathbb{Z}$.

(vi) $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the **Dirichlet function** defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ rational }, \\ 0 & \text{if } x \text{ irrational}. \end{cases}$.
Then f is discontinuous at every $c \in \mathbb{R}$.

(vii) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be the **Thomae function** defined by
$$f(x) = \begin{cases} \dfrac{1}{q} & \text{if } x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z}, q > 0, \gcd(p, q) = 1, \\ 0 & \text{if } x \text{ is an irrational number.} \end{cases}$$
is continuous at every irrational number in $[0, 1]$ and discontinuous at every rational number in $[0, 1]$.

(23) **Relation between the concepts of limit and continuity**:
Let $D \subseteq \mathbb{R}$, and let $c \in \mathbb{R}$ be a limit point of D. Also, let $f : D \longrightarrow \mathbb{R}$ be a function. Then f is continuous at c if and only if $\lim\limits_{x \longrightarrow c} f(x)$ exists and is equal to $f(c)$.
We use the above statement to show that the Thomae function defined above (21: (vii)) is continuous at every irrational number in $[0, 1]$ and discontinuous at every rational number in $[0, 1]$.
First we will show that $\lim\limits_{x \longrightarrow c} f(x) = 0$ for every $c \in \mathbb{R}$.
We use the $\epsilon - \delta$ definition.
Let $\varepsilon > 0$.
We have to find $\delta > 0$, such that $x \in \mathbb{R}, 0 < |x - c| < \delta \Longrightarrow |f(x) - 0| < \varepsilon$.
(Recall that $0 < |x - c| < \delta$ if and only if $x \in (c - \delta, c + \delta) \setminus \{c\}$).
Since $\varepsilon > 0$, by Archimedean Property, there exists $n_0 \in \mathbb{N}$ such that $\dfrac{1}{\varepsilon} < n_0$.
Let $S = \left\{ \dfrac{p}{q} \in [0, 1] : p, q \in \mathbb{Z}, 0 < q < n_0, \gcd(p, q) = 1 \right\}$. Then S is finite.
Choose $\delta > 0$ such that $(c - \delta, c + \delta) \setminus \{c\}$ does not contain any $\dfrac{p}{q} \in S$.
So, $x \in (c - \delta, c + \delta) \setminus \{c\} \Longrightarrow x \neq S$.
Hence $x \in (c - \delta, c + \delta) \setminus \{c\}$ implies either x is irrational or $x = \dfrac{p}{q}, q > n_0$.
Thus $x \in (c - \delta, c + \delta) \setminus \{c\} \Longrightarrow f(x) = 0$ or $f(x) = \dfrac{1}{q}$ where $q > n_0$.
This implies, $|f(x) - 0| = 0$ if x irrational or $|f(x) - 0| = \dfrac{1}{q} < \dfrac{1}{n_0}$.

In any case $|f(x)-0| < \varepsilon$ as $0 < \varepsilon$ and $\dfrac{1}{n_0} < \varepsilon$.
Hence $x \in (c-\delta, c+\delta) \setminus \{c\} \Longrightarrow |f(x)-0| < \varepsilon$.
That is, $0 < |x-c| < \delta \Longrightarrow |f(x)-0| < \varepsilon$.
Hence $\lim\limits_{x \longrightarrow c} f(x) = 0$.
Therefore, $\lim\limits_{x \longrightarrow c} f(x) = f(c)$ if and only if $f(c) = 0$.
Hence f is continuous at c if and only if $f(c) = 0$ (by above statement no. (23)).
But from the definition of the function, $f(x) = 0$ if and only if x is an irrational number.
Therefore Hence f is continuous at c if and only if c is an irrational number.
Thus f is continuous only at irrational numbers in $[0,1]$ and discontinuous at every rational number in $[0,1]$.

(24) **Algebra of Continuous functions**: Let $D \subseteq \mathbb{R}, c \in D$ and $f, g : D \longrightarrow \mathbb{R}$ be functions such that f, g are continuous at c. Then

(i) $f+g$ is continuous at c.

(ii) rf is continuous at c for every $r \in \mathbb{R}$.

(iii) fg is continuous at c.

(iv) If $g(x) \neq 0$ for all $x \in D$ then $\dfrac{f}{g} : D \longrightarrow \mathbb{R}$ is continuous at c.

(25) **Removable discontinuity**: Let $f : \mathbb{R} \to \mathbb{R}$ be a function. A point $c \in \mathbb{R}$ is said to be removable discontinuity of f if $\lim\limits_{x \to c} f(x)$ exists but it is not equal to $f(c)$.

(26) **Essential discontinuity**: Let $f : \mathbb{R} \to \mathbb{R}$ be a function. A point $c \in \mathbb{R}$ is said to be essential discontinuity of f if $\lim\limits_{x \to c} f(x)$ does not exist.

(27) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. $c \in \mathbb{R}$ is called as jump discontinuity of f if both Left hand limit as well as Right hand limit of f exist at c but they are not equal.

3.1.2 Practical 1.1: Limits and Continuity

(A) Objective Questions:

Choose correct alternative in each of the following:

(1) The value of $\lim\limits_{x\to\infty} \dfrac{3x^3+2x}{4x^3-3}$ is

(a) $\frac{1}{3}$ (b) 1 (c) $\frac{3}{4}$ (d) does not exist

(2) The value of $\lim\limits_{x\to 2} \dfrac{x-2}{x^2-4}$ is

(a) $\frac{1}{4}$ (b) 4 (c) $\frac{3}{4}$ (d) does not exist

(3) If $f : \mathbb{R}\setminus\{1\} \longrightarrow \mathbb{R}$ is given by $f(x) := [x] - x$ then $\lim\limits_{x\longrightarrow 1} f(x)$ is

(a) 1 (b) -1 (c) 0 (d) does not exist

(4) If $f(x) = \dfrac{\tan x}{x}$, for $x \neq 0$ then $\lim\limits_{x\to 0} f(x)$

(a) 1 (b) -1 (c) does not exist (d) none of these

(5) If $f : \mathbb{R}\setminus\{1\} \longrightarrow \mathbb{R}$ is given by $f(x) := \dfrac{[x-1]}{x-1}$ then $\lim\limits_{x\longrightarrow 1} f(x)$ is

(a) 1 (b) -1 (c) $\dfrac{x-3}{x^2-2x-3}$ (d) $\dfrac{x+3}{x-3}$

(6) Which of the following functions has a removable discontinuity at $x = 2$?

(a) $\dfrac{\cos(x-2)}{x-2}$ (b) $\dfrac{x+2}{x-2}$ (c) $\dfrac{x^2-x-2}{x-2}$ (d) $\dfrac{1}{\log(x-2)}$

(7) Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \begin{cases} x, & \text{if } x < 0 \\ e^x, & \text{if } x \geq 0 \end{cases}$ then $\lim\limits_{x\to 0^+} f(x)$ is

(a) 0 (b) 1 (c) e (d) does not exist

(8) The value of $\lim\limits_{x\to 0} \dfrac{|x|}{x}$

(a) -1 (b) 1 (c) does not exist (d) None of these

(9) If $f(x) = \begin{cases} 2x+1 & \text{for } x < 0 \\ x-1 & \text{for } x \geq 0 \end{cases}$ then $x = 0$ is a point at which the function f is

(a) discontinuous (b) continuous (c) not defined (d) decreasing

(10) The function $f(x) = |x|$, $x \in \mathbb{R}$ is

(a) continuous
(b) continuous only if $x > 0$
(c) discontinuous
(d) None of these

(11) The function $f(x) = x^2 + x$, $x \in \mathbb{R}$ is

(a) continuous
(b) continuous only if $x > 0$
(c) always negative
(d) None of these

(12) The value of $\lim\limits_{n \longrightarrow \infty} x^n$ for $0 < x < \dfrac{1}{3}$ is

(a) 1 (b) 0 (c) -1 (d) $\dfrac{1}{3}$

(13) The function $f(x) = |x-2| + 3, x \in \mathbb{R}$ is

(a) discontinuous at $x = 3$.
(b) discontinuous at $x = 2$.
(c) continuous everywhere in $\mathbb{R}$.
(d) none of these.

(14) The value of $\lim\limits_{x \longrightarrow 0^+} \dfrac{|x|}{x}$ is

(a) 1 (b) -1 (c) 0 (d) none of these.

(15) Which of the following function has removable discontinuity at $x = 3$

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = \lfloor x \rfloor$.

(b) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = \begin{cases} \dfrac{\sqrt{2x^2-2}-4}{x-3} & \text{if } x \neq 3 \\ 3 & \text{if } x = 3 \end{cases}$.

(c) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$

(d) none of the above.

(16) $\lim\limits_{x \longrightarrow -2} \dfrac{x^2-x-6}{x+2}$ is

(a) 5 (b) 15 (c) -5 (d) None of these.

(17) The function $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = 2x^2 - 8$ is continuous

(a) for each $x \in \mathbb{R}$.
(b) discontinuous at $x = -2$.
(c) discontinuous at $x = 2$.
(d) none of these.

(18) Let $f(x) = \begin{cases} 2x+4 & \text{if } x < 4 \\ 3b & \text{if } x \geq 4. \end{cases}$ If f is continuous on $\mathbb{R}$ then the value of Y is

(a) 2 (b) 4 (c) 3 (d) none of the above.

(19) If $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = 3$ for all $x \in \mathbb{R}$ then $\lim_{x \longrightarrow 7} f(x)$ is

(a) 10 (b) 21 (c) 3 (d) 7

(20) $\lim_{x \longrightarrow p} (\sin x + \cos x)$, where p is a fixed real number is

(a) 1 (b) $\sin p + \cos p$. (c) $\sin p - \cos p$. (d) Doesn't exist.

(21) Consider $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ and $c \in \mathbb{R}$. Under which of the following condition does $\lim_{x \longrightarrow c} f(x)g(x) = 0$?

(a) $\lim_{x \longrightarrow c} f(x) = 0$.
(b) $\lim_{x \longrightarrow c} f(x) = L, L \neq 0$ and g is bounded on $\{x \in \mathbb{R} : 0 < |x - c| < \delta\}$ for some $\delta > 0$
(c) $\lim_{x \longrightarrow c} f(x) = 0$ and g is bounded on $\{x \in \mathbb{R} : 0 < |x - c| < \delta\}$ for some $\delta > 0$.
(d) $\lim_{x \longrightarrow c} f(x) = L, L \neq 0$ and $\lim_{x \longrightarrow c} g(x) = M, M \neq 0$.

(22) $\lim_{x \longrightarrow p} \tan x$, where p is a fixed real number

(a) 0 (b) $\tan p$ (c) may or may not exist. (d) None of these.

(23) $\lim_{x \longrightarrow 0} x \sin\left(\frac{1}{x}\right) =$

3.1

(a) 1 (b) 0 (c) $\dfrac{\pi}{2}$ (d) Doesn't exist.

(24) $\lim\limits_{x\longrightarrow 0} x[x] =$

(a) -1 (b) 0 (c) 1 (d) Doesn't exist.

(25) $\lim\limits_{x\longrightarrow 5} \lfloor x \rfloor =$

(a) 5 (b) 4 (c) 6 (d) Doesn't exist.

(26) $\lim\limits_{x\longrightarrow 1.5} \lfloor x \rfloor =$

(a) 1.5 (b) 0 (c) 1 (d) Doesn't exist.

(27) $\lim\limits_{x\longrightarrow \pi} \lceil x \rceil =$

(a) π (b) 3.14 (c) 4 (d) Doesn't exist.

(28) $\lim\limits_{x\longrightarrow e} \lceil x \rceil =$

(a) e (b) 2.718 (c) 3 (d) Doesn't exist.

(29) $\lim\limits_{x\longrightarrow \frac{\pi}{2}} e^{\cos x} =$

(a) e (b) 0 (c) 1 (d) Doesn't exist.

(30) $\lim\limits_{x\longrightarrow 1} \cos\big(\log(x^2+2x-2)+\sin(x-1)\big) =$

(a) $\frac{\pi}{2}$ (b) 0 (c) 1 (d) Doesn't exist.

(31) $\lim\limits_{x\longrightarrow 0} \dfrac{\lfloor x \rfloor}{x} =$

(a) -1 (b) 0 (c) 1 (d) Doesn't exist.

(32) If $f:\mathbb{R}\longrightarrow\mathbb{R}$, is defined as $f(x)=\begin{cases} x & \text{if } x<0, \\ e^x & \text{if } x\geq 0 \end{cases}$ then $\lim\limits_{x\longrightarrow 0^+} f(x)$

(a) e (b) 0 (c) 1 (d) None of the above.

(33) Consider the following statements:

(i) $\lim_{x \longrightarrow a} f(x) = L \Longrightarrow f(a) = L$.

(ii) $\lim_{x \longrightarrow a} f(x) = L \Longrightarrow f(a)$ is a real number.

(a) Only (i) is true.
(b) Only (ii) is true.
(c) Either (i) or (ii) is true.
(d) Neither (i) nor (ii) is true.

(34) Consider the following statements:

(i) $\lim_{x \longrightarrow a} f(x) = L \Longrightarrow f(a) = L$.

(ii) $f(a) = L \Longrightarrow \lim_{x \longrightarrow a} f(x)$ is a real number.

(a) Only (i) is true.
(b) Only (ii) is true.
(c) Either (i) or (ii) is true.
(d) Neither (i) nor (ii) is true.

(35) If $\lim_{x \longrightarrow a} f(x) = L$ and $\lim_{x \longrightarrow a} g(x) = M$, then $\lim_{x \longrightarrow a} \left(\frac{f(x)}{g(x)} \right) = \frac{L}{M}$ is

(a) LM (b) $\frac{L}{M}$ (c) $\frac{M}{L}$ (d) $\frac{L}{M}$, if $M \neq 0$.

(36) If $f : \mathbb{R} \longrightarrow \mathbb{R}$, is defined as $f(x) = \begin{cases} x^2 & \text{if } x > 2, \\ x^3 & \text{if } x \leq 2 \end{cases}$

then f is continuous ———

(a) everywhere (b) at 2. (c) on $\mathbb{R} \setminus \{2\}$. (d) None of the above.

(37) The function defined by $f(x) = \begin{cases} \sin x & \text{if } x \geq \frac{\pi}{4}, \\ \cos x & \text{if } x < \frac{\pi}{4} \end{cases}$

then f is continuous ———

(a) everywhere. (b) only at $x = \frac{\pi}{4}$. (c) on $\mathbb{R} \setminus \left\{ \frac{\pi}{4} \right\}$. (d) Nowhere.

(38) The function defined by $f(x) = \begin{cases} x^2 + 1 & \text{if } x \neq 5, \\ \alpha & \text{if } x = 5 \end{cases}$

is continuous at 5, ———

(a) for any $\alpha \in \mathbb{R}$. (b) if $\alpha = 6$. (c) if $\alpha = 26$. (d) None of these.

(39) The function defined by $f(x) = \begin{cases} \cos x + 1 & \text{if } x \neq \pi, \\ \alpha & \text{if } x = \pi \end{cases}$ is continuous at π, ———

(a) for any $\alpha \in \mathbb{R}$. (b) if $\alpha = 1$. (c) if $\alpha = \pi + 1$. (d) None of these.

(40) The function defined by $f(x) = \begin{cases} x^2 & \text{if } x > 0, \\ 1 & \text{if } x = 0 \\ x^3 & \text{if } x < 0 \end{cases}$ then f is continuous at 5, ———

(a) only at $x = 0$. (b) at every $x \in \mathbb{R}$. (c) on $\mathbb{R} \setminus \{0\}$. (d) nowhere.

(41) The function defined by $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ is continuous ———

(a) at every rational number only.
(c) at every irrational number only.
(b) only at 0
(d) Nowhere.

(42) Amongst the following, the false statement is ———

(a) There exists a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, which is discontinous only at one point.
(b) There exists a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, which is continous only at one point.
(c) At least one of (a) and (b) is true.
(d) At least one of (a) and (b) is false.

(43) 3 is a removable discontinuity of

(a) $\dfrac{x^2 + 2x - 3}{x - 3}$ (b) $\dfrac{1}{\sin(x - 3)}$ (c) $\dfrac{x + 3}{x - 3}$ (d) $\dfrac{x - 3}{x^2 - 2x - 3}$

(44) Which of the following function has a removable discontinuity at $x = 2$?

(a) $\dfrac{\cos(x - 2)}{x - 2}$ (b) $\dfrac{x + 2}{x - 2}$ (c) $\dfrac{1}{\log(x - 2)}$ (d) $\dfrac{x^2 - x - 2}{x - 2}$

(B) Descriptive Questions

(1) Find the value of δ so that

(i) $0 < |x - 2| < \delta \Longrightarrow |f(x) - 2| < 0.1$, where $f(x) = x$.
(ii) $0 < |x - 2| < \delta \Longrightarrow |f(x) - 11| < 0.5$, where $f(x) = 3x + 5$.
(iii) $0 < |x - 2| < \delta \Longrightarrow |f(x) + 4| < 0.1$, where $f(x) = 7x - 18$.

(2) Use $\epsilon - \delta$ definition to prove the following.

(i) $\lim_{x\longrightarrow 3}(2x+3)=9$

(ii) $\lim_{x\longrightarrow 1}(7-3x)=4$

(iii) $\lim_{x\longrightarrow 2}x^2=4$

(iv) $\lim_{x\longrightarrow 4}\sqrt{x}=2$

(v) $\lim_{x\longrightarrow p}\cos x = \cos p, p \in \mathbb{R}$.

(3) Given that $1-\frac{x^2}{4} \leq h(x) \leq 1+\frac{x^2}{4}$, for all $x \neq 0$. Use Sandwich Theorem to find $\lim_{x\to 0} h(x)$.

(4) It can be shown that the inequalities $1-\frac{x^2}{6} < \frac{x\sin\ x}{2-2\cos\ x} < 1$ hold for all values of x close to zero. Use Sandwich Theorem to find $\lim_{x\to 0}\frac{x\sin x}{2-2\cos x}$.

(5) Using Sandwich Theorem show that $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)=0$.

(6) Suppose $g:\mathbb{R}\longrightarrow\mathbb{R}$ is such that $\frac{\sin x}{x} \leq g(x) \leq 1$ for every $x \in \mathbb{R}\setminus\{0\}$. Show that $\lim_{x\longrightarrow 0} g(x)=1$, using Sandwich Theorem.

(7) $f:\mathbb{R}\longrightarrow\mathbb{R}$ is defined as follows. Discuss continuity of f at $x=1$.

(i) $f(x)=x^2$.

(ii) $f(x)=2x-6$

(iii) $f(x)=x^2+3x+5$

(iv) $f(x)=\begin{cases} x & \text{if } x\geq 1,\\ 1 & \text{if } x<1\end{cases}$

(v) $f(x)=\begin{cases} x^2 & \text{if } x\geq 1,\\ x^3 & \text{if } x<1\end{cases}$.

(vi) $f(x)=\begin{cases} 3x & \text{if } x>1,\\ 2x^2+1 & \text{if } x\leq 1\end{cases}$.

(vii) $f(x)=\begin{cases} 2x+6 & \text{if } x>1,\\ 7 & \text{if } x=1\\ x^2+7 & \text{if } x<1\end{cases}$.

(viii) $f(x)=\begin{cases} 3x-2 & \text{if } x>1,\\ 1 & \text{if } x=1\\ 2x+3 & \text{if } x<1\end{cases}$.

(ix) $f(x)=\begin{cases} \lfloor x\rfloor & \text{if } x>1,\\ 1 & \text{if } x=1\\ \lceil x\rceil & \text{if } x<1\end{cases}$.

(8) $f:\mathbb{R}\longrightarrow\mathbb{R}$ is defined as follows. Find α if f is continuous at the given point a.

(i) $f(x)=\begin{cases} \sin x & \text{if } x\geq \frac{\pi}{4},\\ \alpha^2\cos x & \text{if } x<\frac{\pi}{4}\end{cases}, a=\frac{\pi}{4}$

(ii) $f(x)=\begin{cases} \cos x & \text{if } x\geq 0,\\ x+\alpha & \text{if } x<0\end{cases}, a=0.$

(iii) $f(x)=\begin{cases} \log x & \text{if } x\geq 1,\\ x+\alpha^2 & \text{if } x<1\end{cases}, a=1.$

(iv) $f(x)=\begin{cases} x^2+3x+2 & \text{if } x\geq 1,\\ \alpha^2 x+4 & \text{if } x<1\end{cases}, a=1.$

(v) $f(x)=\begin{cases} 2x+3 & \text{if } x\geq 3,\\ \alpha^2 & \text{if } x=3\\ 6-\alpha & \text{if } x<3\end{cases}, a=3.$

(9) Show that the function $f(x)=\begin{cases} 3-2x & \text{if } x>3,\\ \alpha^2 & \text{if } x\leq 3\end{cases}$ remains discontinuous at 3 for any $\alpha\in\mathbb{R}$.

(10) Show that the function $f(x)=\begin{cases} 3+x & \text{if } x>1,\\ \sin\alpha & \text{if } x\leq 1\end{cases}$ remains discontinuous at 1 for any $\alpha\in\mathbb{R}$.

3.1

(11) Show that the function $f(x) = \begin{cases} x-3 & \text{if } x > 2, \\ \alpha & \text{if } x = 2, \\ 6+x & \text{if } x < 2 \end{cases}$ remains discontinuous at 2 for any $\alpha \in \mathbb{R}$.

(12) Find the maximum value of $r > 0$, such that $f(x) = x^2-4$ remains negative on $(1-r, 1+r)$.

(13) Find the maximum value of $r > 0$ such that the function $f(x) = 9 - x^2$ remains positive on $(2 - r, 2 + r)$.

(14) State examples for each of the following.

(i) A function f such that $\lim\limits_{x\longrightarrow p} |f(x)|$ exists but $\lim\limits_{x\longrightarrow p} f(x)$ does not exist.

(ii) Functions f, g such that $\lim\limits_{x\longrightarrow p} f(x)$ and $\lim\limits_{x\longrightarrow p} g(x)$ exist but $\lim\limits_{x\longrightarrow p} \dfrac{f(x)}{g(x)}$ does not exist.

(iii) Functions f, g such that neither is continuous any where in $\mathbb{R}$ but the function $f + g$ is continuous everywhere in $\mathbb{R}$.

xxxxxxxxxx

3.2 Practical 1.2: Algebra of limits and continuity, Sequential Continuity, Intermediate Value Theorem, Bolzano-Weierstrass Theorem

3.2.1 Prerequisite of Practical 1.2

(1) **Sequential Continuity**:

Let $D \subseteq \mathbb{R}, c \in D$ and let $f : D \to \mathbb{R}$ be a function. Then f is continuous at c if and only if for each sequence (x_n) in D converging to c, the sequence $(f(x_n))$ converges to $f(c)$.

(i) We will show that the **Dirichlet function** (given in the prerequisite no. (21) (vi) of practical number 1 of the second semester) is not continuous at any real number. The function is defined as follows:
$f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$
Using sequential continuity, we can show that f is discontinuous at every $c \in \mathbb{R}$.
Suppose $c \in \mathbb{R}$ be such that f is continuous at c.
By prerequisite of practical no. 1.4 (17) of semester I, there is a sequence of rational numbers say $(x_n) \longrightarrow c$.
As f is continuous at $c, x_n \longrightarrow c \Longrightarrow f(x_n) \longrightarrow f(c)$.
Now, $f(x_n) = 1$ for all $n \in \mathbb{N}$. (as (x_n) is a sequence of rationals).
So, $f(x_n) \longrightarrow 1$.
But $f(x_n) \longrightarrow f(c)$ (as f is continuous at c).
Since a convergent sequence converges to a unique limit, $f(c) = 1$. $(*)$
Now, by prerequisite of practical no. 1.4 (18) of semester I, there is a sequence of irrational numbers say $(y_n) \longrightarrow c$.

Let (y_n) be a sequence of irrational numbers such that $y_n \longrightarrow c$.
As f is continuous at $c, y_n \longrightarrow c \Longrightarrow f(y_n) \longrightarrow f(c)$.
Now, $f(y_n) = 0$ for all $n \in \mathbb{N}$. (as (y_n) is a sequence of irrationals).
So, $f(y_n) \longrightarrow 0$.
But $f(y_n) \longrightarrow f(c)$ (as f is continuous at c).
Since a convergent sequence converges to a unique limit, $f(c) = 0$. $(**)$
Hence, from $(*)$ and $(**), 1 = 0$. This is a contradiction.
Thus if $c \in \mathbb{R}$ then f is not continuous at c.
Hence f is discontinuous at every $c \in \mathbb{R}$.

(ii) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the defined by $f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ -x & \text{if } x \text{ is irrational.} \end{cases}$

This function is given in the prerequisite no. (21-iv) of practical 1 of semester II paper I.
Using sequential continuity, we can show that f is continuous only at 0.
Let $c \in \mathbb{R}$ be such that f is continuous at c.
Let (x_n) be a sequence of rational numbers such that $x_n \longrightarrow c$.(by prerequisite of practical 1.4:(17) of semester I)
As f is continuous at $c, x_n \longrightarrow c \Longrightarrow f(x_n) \longrightarrow f(c)$.
Now, $f(x_n) = x_n$ for all $n \in \mathbb{N}$. (as (x_n) is a sequence of rationals).
So, $f(x_n) \longrightarrow c$ (as $x_n \longrightarrow c$).
But $f(x_n) \longrightarrow f(c)$ (as f is continuous at c).
Since a convergent sequence converges to a unique limit, $f(c) = c$. $(*)$
Let (y_n) be a sequence of irrational numbers such that $y_n \longrightarrow c$.(by prerequisite of practical 1.4:(18) of semester I)

As f is continuous at $c, y_n \longrightarrow c \Longrightarrow f(y_n) \longrightarrow f(c)$.
Now, $f(y_n) = -y_n$ for all $n \in \mathbb{N}$. (as (y_n) is a sequence of irrationals).
So, $f(y_n) \longrightarrow -c$ (as $y_n \longrightarrow c \Longrightarrow -y_n \longrightarrow -c$).
But $f(y_n) \longrightarrow f(c)$ (as f is continuous at c).
Since a convergent sequence converges to a unique limit, $f(c) = -c$. $(**)$
Hence, from $(*)$ and $(**), c = -c$. This implies $2c = 0$ and hence $c = 0$.
Thus if f is continuous at c then $c = 0$.
Hence f is continuous only at c. .

(2) **Intermediate Value Property:** Let I be an interval and $f : I \longrightarrow \mathbb{R}$ be a function. We say that f has the Intermediate Value Property (in short, f has IVP) on I if for all $a, b \in I$ with $a < b$ and $r \in \mathbb{R}$,

$$r \text{ lies between } f(a) \text{ and } f(b) \Longrightarrow r = f(c) \text{ for some } c \in (a, b).$$

(3) **Intermediate Value Theorem**: Let I be an interval and $f : I \longrightarrow \mathbb{R}$ be a continuous function. Then f has the IVP on I i.e. for any $a, b \in I, a < b$ and $r \in \mathbb{R}, r$ lies between $f(a)$ and $f(b)$ then there exists $c \in (a, b)$ such that $r = f(c)$. In particular, $f(I)$ is an interval.

(4) **Bolzano-Weierstrass Theorem**: Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then

(i) f is a bounded function, and

(ii) f attains its bounds on $[a, b]$, that is, there are $r, s \in [a, b]$ such that

$$f(r) = \inf\{f(x) : x \in [a, b]\} \text{ and } f(s) = \sup\{f(x) : x \in [a, b]\}.$$

3.2.2 PRACTICAL 1.2

(A) Objective Questions:

Choose correct alternative in each of the following:

(1) If $\lim\limits_{x \longrightarrow 6} f(x) = 8$ and $\lim\limits_{x \longrightarrow 6} g(x) = -9$ then $\lim\limits_{x \longrightarrow 6} \dfrac{7\sqrt[3]{f(x)} - 6g(x)}{7 + g(x)} =$

(a) 20 (b) -55 (c) -34 (d) -41

(2) $\lim\limits_{x \longrightarrow 0} \dfrac{|x+1| + |x-1| - 2}{x} =$

(a) -1 (b) 1 (c) 0 (d) does not exist.

(3) If $\lim\limits_{x \longrightarrow 0} xf(x) = 3$ then $\lim\limits_{x \longrightarrow 0} \dfrac{f(x)}{|x|} =$

(a) 3 (b) -3 (c) 1 (d) does not exist.

(4) $\lim\limits_{x \longrightarrow 0} \dfrac{(\sqrt{(3+x)} - \sqrt{3})}{x} =$

(a) $\sqrt{3}$ (b) 0 (c) $\dfrac{1}{2\sqrt{3}}$ (d) does not exist.

(5) Suppose $g(x) \leq f(x) \leq h(x)$ for all values of x. If $g(x) = -x^2 - 1, h(x) = \cos 3x$ then $\lim\limits_{x \longrightarrow 0} f(x) =$

(a) 1 (b) -1 (c) 0 (d) can not be determined.

(6) If $\lim\limits_{x \longrightarrow a} f(x) = L$, which one of the following expression is necessarily true?

(a) f is continuous at $x = a$
(b) $f(a)$ does not exist.
(c) $f(a) = L$
(d) $\lim\limits_{x \longrightarrow a^+} f(x) = L$.

(7) The function $f(x) = \dfrac{(7x-1)}{(x^3 - 4x)}$ is continuous on

(a) $\left(-\infty, \frac{1}{7}\right) \cup \left(\frac{1}{7}, \infty\right)$
(b) $(-\infty, 2) \cup (-2, 2) \cup (2, \infty)$
(c) $(-\infty, -2) \cup (-2, 0) \cup (0, 2)\ (2, \infty)$
(d) $(-\infty, 0) \cup (0, \infty)$ does not exist.

(8) Suppose $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous function and takes only rational values. If $f(0) = 3$ then $f(2) =$

(a) 0 (b) 1 (c) -3 (d) none of the above.

(9) The number of points where $g \circ f$ is discontinuous given that $f(x) = \dfrac{1}{(x-3)}$ and $g(x) = \dfrac{1}{(x^2+x-1)}$ is

(a) 1 (b) 2 (c) 3 (d) 4.

(10) Let $f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ 2x + 4 & \text{if } x \geq 2 \end{cases}$.
For what value of the constant c is the function f continuous on $\mathbb{R}$?

(a) 4 (b) 2 (c) 0 (d) 1

(11) Let $f(x) = \begin{cases} \dfrac{x - |x|}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$.
Then f is

(a) continuous everywhere.
(b) continuous for all x except $x = 1$.
(c) continuous for all x except $x = 0$.
(d) none of these.

(12) Let $f(x) = \dfrac{1}{(x^2+1)}$ for all $x \in \mathbb{R}$ then f is

(a) bounded below.
(b) bounded above.
(c) bounded.
(d) neither bounded below nor bounded above.

(13) Let $f(x) = \dfrac{x}{(x^2+1)}$, for all $x \in \mathbb{R}$ then

(a) f is bounded and attains both its bounds.
(b) f is bounded and attains only infimum of f.
(c) f is bounded and attains only supremum of f.
(d) f is bounded and does not attain any of its bounds.

(14) $\lim\limits_{x \longrightarrow \infty} \dfrac{x+4}{3x^2+2} =$

(a) $\frac{1}{3}$ (b) 0 (c) $\frac{4}{3}$ (d) does not exist.

(15) $\lim\limits_{x \longrightarrow \infty} x^2 \sin\left(\dfrac{1}{x}\right) =$

(a) 1 (b) 0 (c) ∞ (d) none of the above.

(B) Descriptive Questions

3.2

(1) Show that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is not continuous at $p = 0$, where

$$f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

.

(2) Show that $\lim\limits_{x \longrightarrow 1} \dfrac{2x^2 + 4x - 6}{x - 1} = 8$ using $\varepsilon - \delta$ definition.

(3) Find constants a and b such that $\lim\limits_{x \longrightarrow 0} \dfrac{\sqrt{ax + b} - 2}{x} = 1$.

(4) Find $\lim\limits_{x \longrightarrow 1.5^+} \dfrac{2x^2 - 3x}{|2x - 3|}$ and $\lim\limits_{x \longrightarrow 1.5^-} \dfrac{2x^2 - 3x}{|2x - 3|}$. Does $\lim\limits_{x \longrightarrow 1.5} \dfrac{2x^2 - 3x}{|2x - 3|}$ exist? Justify your answer.

(5) Is there a number Y such that $\lim\limits_{x \longrightarrow -2} \dfrac{bx^2 + 15x + 15 + b}{x^2 + x - 2}$ exists? If so, find the value of Y and the value of the limit.

(6) Let I be an open interval in $\mathbb{R}$ such that $4 \in I$ and let $f : I \setminus \{4\} \longrightarrow \mathbb{R}$ be a function. Evaluate $\lim\limits_{x \longrightarrow 4} f(x)$ where $x + 2 \leq f(x) \leq x^2 - 10$ for all $x \in I \setminus \{4\}$.

(7) Use Sandwich Theorem to show that $\lim\limits_{x \longrightarrow 0^+} \sqrt{x} e^{\sin\left(\frac{1}{x}\right)}$.

(8) Compute the following limits.

(i) $\lim\limits_{x \longrightarrow 1} \dfrac{\ln x}{\sin(\pi x)}$

(ii) $\lim\limits_{x \longrightarrow \infty} \dfrac{x}{\sqrt{x^2 + 1}}$

(iii) $\lim\limits_{x \longrightarrow 0} (1 - 2x)^{\frac{1}{x}}$

(iv) $\lim\limits_{x \longrightarrow 0} \dfrac{(\sin x)^{100}}{x^{99} \sin 2x}$

(v) $\lim\limits_{x \longrightarrow \infty} (\sqrt{x^2 + x} - x)$

(vi) $\lim\limits_{x \longrightarrow \infty} \dfrac{x^2}{e^x}$

(vii) $\lim\limits_{x \longrightarrow 1} \dfrac{\sin(x - 1)}{x^2 + x - 2}$

(viii) $\lim\limits_{x \longrightarrow 4} \dfrac{1}{\sqrt{x} - 2} - \dfrac{4}{x - 4}$

(ix) $\lim\limits_{x \longrightarrow 1} \dfrac{x^2 - 1}{e^{1 - x^2} - 1}$

(x) $\lim\limits_{x \longrightarrow \infty} \dfrac{\sqrt{x^2 + 4x}}{4x + 1}$

(xi) $\lim\limits_{x \longrightarrow -\infty} (x + \sqrt{x^2 + 2x}$

(xii) $\lim\limits_{x \longrightarrow -\infty} \dfrac{\sqrt{x^2 + 4x}}{4x + 1}$

(9) $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Show that f is continuous at 0 using $\epsilon - \delta$ definition.

(10) If $f(x) = \sqrt{x}$, for all $x > 0$, show that f is continuous at 0 using sequential criterion for continuity.

(11) Let $f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ Show that f is not continuous using sequential criterion for continuity.

(12) Let $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ Show that f is not continuous at any point using sequential criterion for continuity.

(13) Let $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 1-x & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$ Show that f is continuous only at $x = \dfrac{1}{2}$ using sequential criterion for continuity.

(14) $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are functions and $p \in \mathbb{R}$. Prove or disprove the following.

(i) If $f + g$ is continuous at p then either f or g is continuous at p.

(ii) If $f * g$ is continuous at p then either f or g is continuous at p.

(iii) If $|f|$ is continuous at p then either f is continuous at p.

(15) Give an example of a function in each of the following examples.

(i) A continuous but not bounded function on (a, b).

(ii) A continuous bounded function that attains its infimum but does not attain its supremum.

(iii) A continuous bounded function that attains its supremum but does not attain its infimum.

(16) Consider $f : [0,1] \to [0,1]$ defined as $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 1-x & \text{if } x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$
Show that $f(I)$ is an interval. Also show that f does not have Intermediate Value Property.

(17) Use the Intermediate Value Theorem in each of the following examples.

(i) Show that the equation $x^3 - 15x + 1 = 0$ has 3 solutions in the interval $[-4, 4]$.

(ii) Show that the equation $x^2 = \sqrt{x+1}$ has a solution in $[1, 2]$.

(iii) Show that the function $f(x) = (x-a)^2(x-b)^2 + x$ takes the value $\dfrac{a+b}{2}$ for some x.

(iv) Prove that if f, g are continuous on $[a, b]$ and $f(a) > g(a)$ and $f(b) < g(b)$ then there is a point $c \in (a, b)$ such that $f(c) = g(c)$.

(v) Show that there is a square whose diagonal has length between r and $2r$ and has area equal to half the area of the circle of radius r.

(vi) Show that there is a right circular cylinder of height h and radius less than r whose volume is equal to that of the right circular cone of height h and radius r.

(vii) Show that $2^x = \dfrac{10}{x}$ for some $x > 0$.

(viii) Show that $2^x = \dfrac{10}{x}$ has no solution for $x < 0$.

(ix) Show that $2^{3^x} = 10$ for some $x > 0$.

(18) Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a continuous functions such that $f(0) = f(1)$. Prove that there exists $c \in \left[0, \frac{1}{2}\right]$ such that $f(c) = f\left(c + \frac{1}{2}\right)$. [Hint: Define $g(x) = f(x) - f(x + \frac{1}{2})$]

(19) By using Q. 18, show that at any given time there are antipodal points on the equator of the earth that have the same temperature. (The antipode of any place on the Earth is the point on the Earth's surface which is diametrically opposite to it).

(20) Give an example of a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that f is continuous everywhere in $\mathbb{R}$ except at $x = 3$ where it has a removable discontinuity.

3.2

(21) Show that there does not exist a continuous onto function form $[0, 1]$ to $(0, \infty)$.

(22) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous functions. Suppose $f(r) = g(r)$ for all $r \in \mathbb{Q}$ then show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

(23) Fill in the blanks by choosing correct alternative from: (root, negative, $f(2)$, positive, $(-2, 0)$, $(-2, 2)$, continuous)
Consider the function $f(x) = x^2 - 2$, defined on the set of real numbers. The function is -------------------- on $\mathbb{R}$. Therefore, in particular it is continuous on ————-. Now $f(-2)$ is ———— and $f(0)$ is ————-. Therefore, by Intermediate Value Theorem, f has a root in ————. Also, ———— is positive. Therefore, f has a ———— in $(0, 2)$.

(24) Consider the function $f(x) = x^3 - 15x + 1$. Show that this function has roots in $(-4, 3)$, $(1, 2)$ and $(3, 4)$.

(25) Show that the above function, remains positive for all $x > 4$ and remains negative for all $x < -4$.

(26) Fill in the blanks by choosing correct alternative from: (cannot be, need not be, may be, must be)
Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function. Then $\{f(x) : x \in [a, b]\}$ ———— an interval, but ———— equal to $[f(a), f(b)]$. Further, if $\{f(x) : x \in [a, b]\}$ is not an interval, then $f : [a, b] \longrightarrow \mathbb{R}$———— a continuous function. Finally, if $\{f(x) : x \in [a, b]\}$ is an interval, then $f : [a, b] \longrightarrow \mathbb{R}$ may or———— a continuous function.

(27) State true or false. If the given statement is false, justify it by giving a counter example.

(i) If f is differentiable at p, then f is continuous at p.
(ii) If f is not differentiable at p, then f is not continuous at p.
(iii) If f is not continuous at p, then f is not differentiable at p.
(iv) If f is continuous at p, then f is differentiable at p.

(28) Give an example of a function, other than the modulus function, which is continuous but not differentiable on an interval.

xxxxxxxxxxxxx

3.3 Practical 1.3: Properties of Differentiable Functions, Differentiability of: Inverse functions, Composite functions, Implicit functions

3.3.1 Prerequisite of Practical 1.3

(1) Let $D \subseteq \mathbb{R}$ and $c \in D$ be an interior point of D. A function $f : D \longrightarrow \mathbb{R}$ is said to be **differentiable** at c if the limit $\lim\limits_{h \longrightarrow 0} \dfrac{f(c + h) - f(c)}{h}$, that is, $\lim\limits_{x \longrightarrow c} \dfrac{f(x) - f(c)}{x - c}$ exists.

In this case, the value of the limit is denoted by $f'(c)$ and is called the **derivative** of f at c.

If every point of D is an interior point of D and f is differentiable at every point of D, then f is said to be **differentiable** on D.
If f is differentiable on D, we obtain a new function from D to $\mathbb{R}$ whose value at $c \in D$ is $f'(c)$. This function is denoted by f' and is called the **derivative** of f.
Other notations: $\dfrac{df}{dx}$ or $\dfrac{dy}{dx}$ if $y = f(x)$.
$f'(c)$ is denoted by $\left.\dfrac{df}{dx}\right|_{x=c}$ or $\left.\dfrac{dy}{dx}\right|_{x=c}$

(2) **Geometric interpretation of derivative of a function at a point c :**
The given function is differentiable at c if the graph of the function has a unique non-vertical tangent at c.

(3) Let $D \subseteq \mathbb{R}$ and let c be an interior point of D. If $f : D \longrightarrow \mathbb{R}$ is differentiable at c then f is continuous at c. Converse not true.

(4) Some differentiable functions.

(i) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a constant function, then $f'(c) = 0$ for each $c \in \mathbb{R}$.
That is if f is a constant function then f' is identically 0.
Consider $f : (-\infty, 1) \cup (1, \infty) \longrightarrow \mathbb{R}, f(x) = \begin{cases} 2 & \text{if } x \in (-\infty, 1), \\ 3 & \text{if } x \in (1, \infty). \end{cases}$
We can check the following:
(a) f is not a constant function.
(b) f is differentiable on $\mathbb{R} \setminus \{1\}$.
(c) $f'(c) = 0$ for all $c \in \mathbb{R} \setminus \{1\}$.
(d) So, f is a non-constant function having $f'(c) = 0$ for all $c \in (-\infty, 1) \cup (1, \infty)$.

(ii) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = 1$ for all $x \in \mathbb{R}$. Then f is differentiable on $\mathbb{R}$ and $f'(x) = 1$ for all $x \in \mathbb{R}$.

(iii) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = |x|$ for all $x \in \mathbb{R}$. Then f is differentiable at $c \in \mathbb{R} \setminus \{0\}$.

(iv) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^{\frac{2}{3}}$ or all $x \in \mathbb{R}$. Then f is differentiable at $c \in \mathbb{R} \setminus \{0\}$.

(5) **Left (hand) derivative** : Let $D \subseteq \mathbb{R}$ and $c \in D$ be such that $(c - r, c] \subseteq D$ for some $r > 0$. Also let $f : D \longrightarrow \mathbb{R}$ be a function. If the left limit $\lim\limits_{x \longrightarrow c^-} \dfrac{f(x) - f(c)}{x - c}$ exists, then it is called the **left (hand) derivative** of f at c and is denoted by $f'_-(c)$.

(6) **Right (hand) derivative** : Let $D \subseteq \mathbb{R}$ and $c \in D$ be such that $[c, c + r) \subseteq D$ for some $r > 0$. Also let $F : D \longrightarrow \mathbb{R}$ be a function. If the right limit $\lim\limits_{x \longrightarrow c^+} \dfrac{f(x) - f(c)}{x - c}$ exists, then it is called the **right (hand) derivative** of f at c and is denoted by $f'_+(c)$.

(7) If c is an interior point of D, then we can find both, $f'_-(c), f'_+(c)$. It follows from statement (13) of prerequisite of practical 1.1, that f is differentiable at c if and only if both $f'_-(c)$ and $f'_+(c)$ exist and are equal.

(8) Some non-differentiable functions.

3.3

(i) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = |x|$ for all $x \in \mathbb{R}$. Then f is not differentiable at $x = 0$ as both $f'_-(c)$ and $f'_+(c)$ exist but are not equal.

(ii) $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = x^{\frac{2}{3}}$ or all $x \in \mathbb{R}$. Then f is not differentiable at $x = 0$ as neither $f'_-(c)$ nor $f'_+(c)$ exists.

(9) $f : [a, b] \longrightarrow \mathbb{R}$ is differentiable on $[a, b]$, if f is differentiable at every point of (a, b) and if $f'_+(a)$ and $f'_-(b)$ exist.

(10) The following statement is very useful to prove a number of basic properties of derivatives. **Caratheodory Lemma**: Let $D \subseteq \mathbb{R}$ and let $c \in D$ be an interior point of D. Then a function f is differentiable at c if and only if there exists a fnction $f_1 : D \longrightarrow \mathbb{R}$ such that $f(x) - f(c) = (x - c)f_1(x))$ for all $x \in D$, and f_1 is continuous at c. Moreover, if these conditions hold, then $f'(c) = f_1(c)$.

(11) **Algebra of Differentiable Functions**: Let $D \subseteq \mathbb{R}$, let c be an interior point of D and $f, g : D \longrightarrow \mathbb{R}$ be differentiable at c. Then

(i) $f + g$ is differentiable at c and $(f + g)'(c) = f'(c) + g'(c)$,

(ii) rf is differentiable at c and $(rf)'(c) = rf'(c)$ for every $r \in \mathbb{R}$.

(iii) fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

(iv) If $f(c) \neq 0$ then the function $\frac{1}{f}$ is differentiable at c and $\left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{(f(c))^2}$

(v) If $g(c) \neq 0$ then the function $\frac{f}{g}$ is differentiable at c and $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$.

(12) **Chain Rule**: Let $D, E \subseteq \mathbb{R}$ and $f : D \longrightarrow \mathbb{R}, g : E \longrightarrow \mathbb{R}$ be functions such that $f(D) \subseteq E$. Suppose $c \in D$ is such that c is an interior point of D and $f(c)$ is an interior point of E. If f is differentiable at c and g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'\Big(f(c)\Big) \cdot f'(c)$.

(13) **Differentiability of Inverse Function**: Let $I \subseteq \mathbb{R}, I$ be an interval and let $c \in I$ be an interior point of I. Suppose $f : I \longrightarrow \mathbb{R}$ is a one-one and continuous function. Let $f^{-1} : f(I) \longrightarrow I$ be the inverse function. If f is differentiable at c and $f'(c) \neq 0$ then f^{-1} is differentiable at $f(c)$ and

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}.$$

(14) **Implicit functions:** The equations of the form $F(x, y) = 0$, where F is a real valued function defined on some subset E of the plane $\mathbb{R}^2$, and (x, y) varies over the points of E are the plane curves C. We say that C is the implicitly defined curve $F(x, y) = 0, (x, y) \in E$. For example, the circle of radius 5 centered at the origin is the curve $x^2 + y^2 - 25 = 0$. The Chain Rule helps us to find the tangents to implicitly defined curves. Here $F(x, y) = x^2 + y^2 - 25$. To find the tangent at a point $(3, 4)$, we differentiate $F(x, y)$ with respect to x, treating y as a function of x. Using the Chain Rule, we obtain $2x + 2y\frac{dy}{dx} = 0$ and hence $\left(\frac{dy}{dx}\right)_{(3,4)} = \left(-\frac{x}{y}\right)_{(3,4)} = -\frac{3}{4}$.

Hence the equation of the tangent to this circle $F(x,y)=0$ at the point $(3,4)$ is given by the line $y-4=-\frac{3}{4}(x-3)$, that is $3x+4y-25=0$.

3.3.2 PRACTICAL 1.3

(A) Objective Questions:

Choose correct alternative in each of the following:

(1) Which of the following functions is differentiable at $x=0$?

(a) $|x|$

(b) $f(x)=\begin{cases} x\sin\left(\frac{1}{x}\right) & \text{if } x\neq 0, \\ 0 & \text{if } x=0. \end{cases}$

(c) $x^{\frac{1}{3}}$

(d) $f(x)=\begin{cases} x^2 & \text{if } x\in\mathbb{Q}, \\ 0 & \text{if } x\notin\mathbb{Q}. \end{cases}$

(2) Which of the following functions is continuous at $x=0$ but not differentiable at $x=0$?

(a) $x^{-\frac{4}{3}}$.

(b) $x^{-\frac{1}{3}}$.

(c) $x^{\frac{1}{3}}$.

(d) $x^{\frac{4}{3}}$.

(3) Suppose $f(x)=x^4+ax^2+bx$, (where $a,b\in\mathbb{R}$) satisfies the following two conditions, $\lim\limits_{x\longrightarrow 2}\frac{f(x)-f(2)}{x-2}=4$ and $\lim\limits_{x\longrightarrow 1}\frac{f(x)-f(1)}{x^2-1}=9$. Then value of $b-a$ is

(a) 55. (b) 88. (c) 66. (d) 77.

(4) Let $f(x)=\begin{cases} x^2 & \text{if } x\leq 2, \\ 8-2x & \text{if } x>2. \end{cases}$.
Then f is

(a) continuous but not differentiable at $x=2$.
(b) Not continuous at $x=2$ but differentiable at $x=2$.
(c) Differentiable at $x=2$.
(d) Neither continuous nor differentiable at $x=2$.

(5) Let $f(x)=|x-2|+|x-3|$, for all $x\in\mathbb{R}$ then $f'(2)=$

(a) -2. (b) 0. (c) 2. (d) not defined.

(6) $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a differentiable function where the tangent to the graph of $f(x)$ at $x=2$ is $y=x+1$, then

(a) $f(2)=1, f'(2)=1$.

(b) $f(2)=1, f'(2)=0$.

(c) $f(2)=3, f'(2)=1$.

(d) $f(2)=3, f'(2)=0$.

(7) Let $f(x) = \max\{x, x^3\}$, for all $x \in \mathbb{R}$ then the number of points where f is not differentiable are

(a) 0. (b) 1. (c) 2. (d) 3.

(8) Consider the following statements.

(i) if f is continuous at a then f is differentiable at a.

(ii) If $\lim\limits_{x \longrightarrow a} \dfrac{f(x) - f(a)}{x - a}$ exists then f is differentiable at a.

(iii) If $\lim\limits_{x \longrightarrow a} f(x)$ exists then f is continuous at a.

(iv) If f is differentiable at a then $\lim\limits_{x \longrightarrow a} f(x) = f(a)$

(a) only (ii) is true. (c) (i) and (iii) are true.

(b) (ii) and (iii) are true. (d) $(ii), (iii)$ and (i) are true.

(9) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable functions. If $h(x) = f(g(x))$ and $f(-2) = 8, f'(-2) = 4, f'(5) = 3, g(5) = -2, g'(5) = 6$ then $h'(5) =$

(a) 12 (b) 24 (c) 8 (d) -16

(10) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be differentiable functions such that $g(x) < 0$, for all $x \in \mathbb{R}$ and $f(0) = 3$. If $h(x) = f(x) * g(x)$ and $h'(x) = f(x)g'(x)$ then $f(x) =$

(a) 3 (b) $f'(x)$ (c) $g(x)$ (d) e^x

(11) If $f(u) = \sin u$ and $u = g(x) = x^2 - 9$ then $(f \circ g)'(3) =$

(a) 6 (b) 1 (c) 0 (d) 9

(12) If $x = t - \sin t, y = 1 - \cos t$ then $\dfrac{dy}{dx} =$

(a) $\dfrac{\sin t}{1 - \cos t}$ (b) $\dfrac{1 - \cos t}{\sin t}$ (c) $\dfrac{\sin t}{1 + \cos t}$ (d) $\dfrac{1 - x}{y}$

(13) If $x = \cos^3 \theta, y = \sin^3 \theta$ then $\dfrac{dy}{dx} =$

(a) $\tan^3 \theta$ (b) $\tan \theta$ (c) $-\tan \theta$ (d) $\cot \theta$

(14) If $x = \dfrac{1}{1 - t}, y = 1 - \ln(1 - t)$ where $t < 1$ then $\dfrac{dy}{dx} =$

(a) $\dfrac{1}{1 - t}$. (b) $\dfrac{1}{(1 - t)^3}$. (c) $\dfrac{1}{x}$. (d) $\dfrac{(1 - t)^2)}{t}$.

(15) If $x^2 + 2xy = y^2$ then $\dfrac{dy}{dx}$ is (assuming y is a function of x)

(a) $\dfrac{x+y}{x-y}$. (b) $\dfrac{x+y}{y-x}$. (c) $2x+2y$. (d) $\dfrac{x+1}{y}$.

(16) If $x+\cos(x+y)=0$ then $\dfrac{dy}{dx}$ is (assuming y is a function of x)

(a) $\dfrac{x}{\sin(x+y)}$. (b) $\dfrac{1}{\sin(x+y)}$. (c) $\dfrac{1}{\sin(x+y)}-1$. (d) $\dfrac{1-\sin x}{\sin y}$.

(17) If h is the inverse function of f and if $f(x)=\dfrac{1}{x}$ then $h'(3)=$

(a) -9. (b) $-\dfrac{1}{9}$. (c) $\dfrac{1}{9}$. (d) 9.

(18) Suppose $f(x)=2x^3-3x$ and h is the inverse function of f then $h'(-1)=$

(a) -1. (b) $\dfrac{1}{3}$. (c) $-\dfrac{1}{3}$. (d) 1.

(19) the derivative of the inverse function of $f:[0,1]\longrightarrow\mathbb{R}, f(x)=xe^x$ at $x=0.5$ is

(a) $\dfrac{2}{3\sqrt{e}}$. (b) $\dfrac{3}{2\sqrt{e}}$. (c) $\dfrac{2\sqrt{e}}{3}$. (d) None of these.

(20) Let $f,g:\mathbb{R}\longrightarrow\mathbb{R}$ be differentiable functions. If f and g are inverses of each other and $f'(2)=5$ and $g'(2)=$

(a) -5. (c) $-\dfrac{1}{5}$.
(b) $\dfrac{1}{5}$. (d) can not calculate as data is insufficient.

(21) If the equation of a curve is $x^2y^2=x^2-y^2$ then the slope of the tangent to the curve at $(1,1)$ is .

(a) -5. (b) $\dfrac{1}{5}$. (c) 0. (d) none of these.

(22) The slope of the normal to the curve $y=x^3y+2x^2-y^2$ at $(2,-1)$ is

(a) $\dfrac{9}{4}$. (b) $\dfrac{4}{9}$. (c) 0. (d) none of these.

(B) Descriptive Questions

(1) Let f be a function defined on $\mathbb{R}$ by $f(x)=\begin{cases}\sin\left(\dfrac{1}{x}\right) & \text{if } x\neq 0,\\ 0 & \text{if } x=0.\end{cases}$.

Answer the following.

(i) Is f continuous at $x=0$?

(ii) Is f differentiable at $x=0$?

(2) Find the derivative of the following functions using the definition.

3.3

(i) $f(x) = x^2$, for all $x \in \mathbb{R}$

(ii) $f(x) = x^n$, for all $x \in \mathbb{R}$ where $n \in \mathbb{N}$

(iii) $f(x) = \dfrac{1}{x}$ for all $x \in \mathbb{R} \setminus \{0\}$

(iv) $f(x) = \sqrt{x}$ for all $x \in \mathbb{R}$

(v) $f(x) = \dfrac{1}{\sqrt{x}}$ for all $x \in \mathbb{R} \setminus \{0\}$

(vi) $f(x) = \dfrac{x+1}{2-x}$ for all $x \in \mathbb{R} \setminus \{2\}$

(vii) $f(x) = \sqrt{x^3 - x}$ for all $x > 1$

(viii) $f(x) = \sin x$ for all $x \in \mathbb{R}$

(ix) $f(x) = \begin{cases} 2 + \sqrt{x} & \text{if } x \geq 1, \\ x^2 + \dfrac{5}{2} & \text{if } x < 1. \end{cases}$ at $x = 1$.

(3) Check whether the following functions from $\mathbb{R} \longrightarrow \mathbb{R}$ are differentiable at the mentioned point/domain.

(i) $f(x) = \begin{cases} 0 & \text{if } x < 0, \\ x^3 & \text{if } x \geq 0. \end{cases}$ at $x = 0$.

(ii) $f(x) = \begin{cases} 1 & \text{if } x < 0, \\ 2x+1 & \text{if } x \geq 0. \end{cases}$ at $x = 0$.

(iii) $f(x) = \begin{cases} 5x-1 & \text{if } x \leq 3, \\ 6x+2 & \text{if } x > 3. \end{cases}$ at $x = 3$.

(iv) $f(x) = \begin{cases} 2x+5 & \text{if } x < 1, \\ x^3-1 & \text{if } x \geq 1. \end{cases}$ at $x = 1$.

(v) $f(x) = \begin{cases} 4x+1 & \text{if } x \leq 2, \\ x^2+5 & \text{if } x > 2. \end{cases}$ at $x = 2$.

(vi) $f(x) = \begin{cases} 4x+1 & \text{if } x \leq 2, \\ x^2+5 & \text{if } x > 2. \end{cases}$ at $x = 2$.

(vii) $f(x) = \begin{cases} x^2+x+1 & \text{if } x < 1, \\ 5x^4-2 & \text{if } x \geq 1. \end{cases}$ at $x = 1$.

(viii) $f(x) = \begin{cases} 8x-5 & \text{if } x \leq 2, \\ 3x^2-4x+7 & \text{if } x > 2. \end{cases}$ at $x = 2$.

(ix) $f(x) = \begin{cases} x^2-3 & \text{if } x < 1, \\ 3x^2-4x-1 & \text{if } x \geq 1. \end{cases}$ at $x = 1$.

(x) $f(x) = \begin{cases} e^x+3 & \text{if } x \leq 0, \\ \sin x + 4x & \text{if } x > 0. \end{cases}$ at $x = 0$.

(xi) $f(x) = \begin{cases} \cos\left(\frac{1}{x}\right) & \text{if } x \leq 0, \\ 1 & \text{if } x = 0. \end{cases}$ at $x = 0$.

(xii) $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x \leq 3, \\ 1 & \text{if } x = 3. \end{cases}$ at $x = 3$.

(xiii) $f(x) = \begin{cases} x^2+1 & \text{if } x < 2, \\ 6x-4 & \text{if } x \geq 2. \end{cases}$ at $x = 2$.

(xiv) $f(x) = \begin{cases} x^2+x & \text{if } x \in \mathbb{Q}, \\ x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ at $x = 0$.

(xv) $f(x) = \begin{cases} x^2+1 & \text{if } x \in \mathbb{Q}, \\ 2x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$ at $x = 0$.

(xvi) $f(x) = \begin{cases} x \sin\left(\dfrac{1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ at $x = 0$.

(xvii) $f(x) = \begin{cases} x^{-2} \sin\left(x^3\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ at $x = 0$.

(xviii) $f(x) = |x-2|$ at $x = 2$.

(xix) $f(x) = |x-1| + |x-2|$ at $x = 0, 1, 2$

(xx) For $n > 1$, $f(x) = \begin{cases} x^n \sin\left(\dfrac{1}{x^2}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ at $x = 0$.

(4) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function which is differentiable at 0 and $f(0) = 0$.
If $\lim\limits_{h \longrightarrow 0} \dfrac{f(4h) + f(2h) + f(h) + f\left(\frac{h}{2}\right) + f\left(\frac{h}{4}\right) + \cdots}{h} = 64$ then find $f'(0)$.

(5) Let $f : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be a function which satisfies $f(xy) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$. If $f'(1) = 0$, find a solution for $f(x)$.

(6) Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an even function and has a derivative at every point, then the derivative f' is an odd function. Also prove that if $g : \mathbb{R} \longrightarrow \mathbb{R}$ is a differentiable odd function then g' is an even function.

(7) Suppose $|f(x)| \leq x^2$ for $-1 \leq x \leq 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

(8) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded function and $f(x) = x^2 * g(x)$ for all $x \in \mathbb{R}$. Show that f is differentiable at $x = 0$.

(9) $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Define $F(x) = \begin{cases} \dfrac{f(x)\sin^2 x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

Find $F'(0)$.

(10) The function $f(x) = \begin{cases} e^x & \text{if } x \leq 1, \\ mx + b & \text{if } x > 1. \end{cases}$ is differentiable at $x = 1$. Find the values for the constants m and b.

(11) Suppose u and v are functions of x that are differentiable at $x = 0$. If $u(0) = 5, u'(0) = -3, v(0) = -1, v'(0) = 2$, find the values of the following derivatives at $x = 0$.

(i) $\dfrac{d}{dx}(uv)$ (ii) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)$ (iii) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right)$ (iv) $\dfrac{d}{dx}(7v - 2u)$

(12) Suppose that f and g are differentiable functions such that $f(g()) = x$ and $f'(x) = 1 + (f(x))^2$. Show that $g'(x) = \dfrac{1}{1+x^2}$.

(13) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions such that $f'(x) = g(x)$ and $g'(x) = f(x)$ for all $x \in \mathbb{R}$. Prove that $f^2 - g^2$ must be a constant function.

(14) If f, g, h and ϕ are differentiable functions, then state the derivative of $f \circ g(x)$, $((f \circ g) \circ h)(x)$ and $(f \circ g \circ h \circ \phi)(x)$.

(15) Express each of the following functions in the form of $(f \circ g \circ h)(x)$, by clearly stating the functions f, g, h. Hence write the derivative of each.

(i) $\sin(\cos(e^x))$ (ii) $\sqrt{\sin(x^2)}$ (iii) $e^{\tan^{-1}(3x+2)}$

(16) If $-1 \leq x \leq 1$, then prove that the derivative of Evaluate $\dfrac{df}{dx}$ at $x = \dfrac{1}{2}$ and $\dfrac{df^{-1}}{dy}$ at $y = f\left(\dfrac{1}{2}\right)$ and show that at these points. $\dfrac{df^{-1}}{dy} = \dfrac{1}{\dfrac{df}{dx}}$ where $f(x) = sin^{-1}x, -1 \leq x \leq 1, -\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$.

(17) Find $\dfrac{dy}{dx}$ in the following examples where y is a function of x.

(i) $\sin y + x^2y^3 - \cos x = 2y$

(ii) $3xy^2 + \cos y^2 = 2x^3 + 5$

(iii) $5x^2 - x^3 \sin y + 5xy = 10$

(iv) $\tan 5y - y \sin x + 3xy^2 = 9$

3.3

(v) $xe^{x^2+y^2} = 5$

(vi) $x^3 + y^3 = 3axy$

(vii) $x^2y^2 = x^2 - y^2$

(viii) $\cos x^2 = xe^y$

(ix) $y^3 + y^2x + 4x^2 = 6$

(x) $x - \cos x^2 + \dfrac{y^2}{x} + 3x^5 = 4x^3$

(xi) $4x^2 + \sin x * y^4 = 3y$

(xii) $e^{xy} = e^{\cos y}$

(xiii) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$

(xiv) $e^x + e^y = x + y$

(xv) $y \cos x = 1 + \sin xy$

(xvi) $xy = x \sin y = 4$

(18) If $\sec\left(\dfrac{x+y}{x-y}\right) = a$, show that $\dfrac{dy}{dx} = \dfrac{y}{x}$

(19) If $\sin^{-1}\left(\dfrac{x^2-y^2}{x^2+y^2}\right) = \ln a$, show that $\dfrac{dy}{dx} = \dfrac{y}{x}$

(20) If $x^4y^5 = (x+y)^9$, show that $\dfrac{dy}{dx} = \dfrac{y}{x}$

(21) If $x^py^q = (x+y)^{p+q}$, show that $\dfrac{dy}{dx} = \dfrac{y}{x}$

(22) If $y = x \sin y$, show that $\dfrac{dy}{dx} = \dfrac{y}{x(1 - x\cos y)}$

(23) Find the equation of the tangent to the curve $x^3 + y^3 = 6xy$ at the point $(3, 3)$

(24) Find $\dfrac{dy}{dx}$ if $\cos(x+y) = y^2 \sin x$

(25) Show that every curve in the family $xy = c, c \neq 0$ is orthogonal to every curve in the family $x^2 - y^2 = k, k \neq 0$.

xxxxxxxxxxxxxxx

3.4 Practical 1.4: Higher Order Derivatives, Leibniz Theorem

3.4.1 Prerequisite of Practical 1.4

(1) Let $I \subseteq \mathbb{R}$ be an open interval and $c \in I$. Suppose the function $f : I \longrightarrow \mathbb{R}$ is differentiable at every point of I. Then the derivative function f' is defined on I. In case f' is differentiable at c, then we say that f is **twice differentiable** at c and denote the derivative of f' at c by $f''(c)$. The quantity $f''(c)$ is called the **second derivative** or **second -order derivative** of f at c. (note that when we are finding $f''(c)$ we need $f'(c+h)$, that is, we need differentiability of f at $c+h$ also.)
If f' is differentiable at every point in I, then the second derivative function f'' is defined on I. If f'' is also differentiable at c, then we say that f is **thrice differentiable** or **third -order derivative** at c and denote the derivative of f'' at c by $f'''(c)$. Similarly, we can define $n-$ times differentiability of f and the n^{th} derivative or n^{th} **-order derivative** $f^{(n)}(c)$ for every $n \in \mathbb{N}$. The notations $\dfrac{d^2f}{dx^2}\Big|_{x=c}$, $\dfrac{d^3f}{dx^3}\Big|_{x=c}$, and $\dfrac{d^nf}{dx^n}\Big|_{x=c}$ are sometimes used

instead of $f''(c), f'''(c)$, and $f^{(n)}(c)$, respectively.
If f is $n-$times differentiable at c for every $n \in \mathbb{N}$, then f is said to be **infinitely differentiable** at c.

(2) $a, b \in \mathbb{R}, m, n \in \mathbb{N}$

(i) If $y = (ax+b)^m$, then
$$y_n = \begin{cases} {}^mP_n a^n (ax+b)^{m-n} & \forall\, n \le m \\ 0 & \forall\, n > m \end{cases}.$$

(ii) If $y = \dfrac{1}{ax+b}$ then
$$y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}} \quad \forall\, n \in \mathbb{N}.$$

(iii) If $y = \ln(ax+b)$ then
$$y_n = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n} \quad \forall\, n \in \mathbb{N}.$$

(iv) If $y = \sin(ax+b)$ then
$$y_n = a^n \sin\left(ax+b+n\cdot\frac{\pi}{2}\right) \quad \forall\, n \in \mathbb{N}.$$

(v) If $y = \cos(ax+b)$ then
$y_n = a^n \cos(ax+b+n\cdot\frac{\pi}{2})\ \forall\, n \in \mathbb{N}$.

(vi) If $y = e^{mx}$, then $y_n = m^n e^{mx}\ \ \forall\, n \in \mathbb{N}$.

(vii) If $y = a^{mx}, a > 0$ then
$y_n = m^n a^{mx} (\ln a)^n, \quad \forall\, n \in \mathbb{N}$.

(3) **Leibniz Rule** : Let $I \subseteq \mathbb{R}$ be an open interval and $f, g : I \longrightarrow \mathbb{R}$ be functions. If f, g are $n-$times differentiable on I then the product function $f \cdot g : I \longrightarrow \mathbb{R}$ given by $(f\cdot g)(x) = f(x)\cdot g(x)$, for all $x \in I$ is also $n-$times differentiable on I and the n^{th} order derivative of the product $f \cdot g$ at $c \in I$ is given by:
$$(f\cdot g)^{(n)}(c) = {}^nC_0 f^{(n)}(c)g^{(0)}(c) + {}^nC_1 f^{(n-1)}(c)g^{(1)}(c) + \cdots + {}^nC_r f^{(n-r)}(c)g^{(r)}(c) + \cdots + {}^nC_n f^{(0)}(c)g^{(n)}(c)$$
where $f^{(0)}(c) = f(c), g^{(0)}(c) = g(c)$.

(A) Objective Questions:

Choose correct alternative in each of the following:

(1) If $y = (2x+13)^{20}$ then y_{15} at $c = 0$ is

(a) $\dfrac{20!}{5! * 15!}2^{15} * 13$ (b) $\dfrac{20!}{5!}2^{15} * 13^5$ (c) $\dfrac{20!}{5!}2^{15} * 13^{15}$ (d) None of these.

(2) If $y = \dfrac{1}{3x-5}$ for all $x \neq \dfrac{5}{3}$ then y_8 is

(a) $\dfrac{-8!3^8}{(3x-5)^8}$ (b) $\dfrac{8!3^8}{(3x-5)^9}$ (c) $\dfrac{-8!3^8}{(3x-5)^9}$ (d) None of these.

(3) If $y = \ln(6-5x)$ for all $x < \dfrac{6}{5}$ then y_7 is

(a) $\dfrac{6!5^6}{(6-5x)^8}$ (b) $\dfrac{-6!5^6}{(6-5x)^8}$ (c) $\dfrac{-6!5^6}{(6-5x)^9}$ (d) None of these.

(4) If $y = \sin(3x+5)$ then y_{16} is

(a) $3^{16}\cos(3x+5)$ (b) $3^{16}\sin(3x+5)$ (c) $3^{15}\sin(3x+5)$ (d) None of these

(5) If $y = \cos(2x + \dfrac{5\pi}{2})$ then y_{13} is

(a) $2^{13}\cos(2x)$ (b) $-2^{13}\cos(2x)$ (c) $2^{13}\sin(2x)$ (d) None of these

(6) If $y = 4^{9x}$ then y_{33} at $c = 0$ is

(a) $9^{33}4^{9}$ (b) $9^{33}4^{9}$ (c) $9^{32}4^{9}$ (d) None of these

(7) If $y = 3^{2x}$ then y_{15} is

(a) $2^{14}3^{2x}\ln 3$ (b) $2^{15}3^{2x}\ln 3$ (c) $2^{15}3^{2x-1}\ln 3$ (d) None of these.

(8) If $y = e^{-mx}$ then

(a) $y_{n+1} + (m + 1)y_n = 0$ (b) $y_{n+1} + my_n = 0$ (c) $y_{n+1} + (m - 1)y_n = 0$ (d) none of these.

(9) If $y = \sin(m\sin^{-1}x)$ then

(a) $(1 - x^2)y_2 - 2xy_1 + m^2y = 0$
(b) $(1 - x^2)y_2 - xy_1 + m^2y = 0$
(c) $(1 - x^2)y_2 + xy_1 + m^2y = 0$
(d) None of these

(10) If $y = \cos x^2$ then third order derivative $y_3 =$

(a) $8x^3\sin x^2 - 12x\cos x^2$
(b) $4x\cos x^2 - 8x^3\sin x^2$
(c) $8x^3\sin x^2 - 4x\cos x^2$
(d) $12x\cos x^2 - 8x^3\sin x^2$

(B) Descriptive Questions

(1) Given $y = \dfrac{1}{x} + \cos 2x$, find $\dfrac{d^5y}{dx^5}$.

(2) Using the n^{th} order derivatives of standard functions, find the n^{th} order derivatives of the following functions.

(i) $\sin^2 x$
(ii) $\dfrac{1}{1-5+6x^2}$
(iii) $\sin 6x\cos 4x$
(iv) $\sin^2 x\cos^3 x$
(v) $x\ln x$

(3) Use Leibniz Rule to prove the following: (y_n denotes the n^{th} order derivative of y)

(i) If $y = \sin ax + \cos ax$ (where $a \in \mathbb{R}$) then $y_n = a^n[1 + (-1)^n\sin 2ax]^{\frac{1}{2}}$.
(ii) If $y = e^{-mx}$ then $y_{n+1} + my_n = 0$.
(iii) If $y = e^{mx} + e^{-mx}$ then $y_{n+2} + m^2y_n = 0$.
(iv) If $y = x^3e^x$ then
 (a) $y_3 - 3y_2 + 3y_1 - y = 6e^x$
 (b) $y_{n+3} - 3y_{n+2} + 3y_{n+1} - y = 6e^x$.

(v) If $y = a\cos(\ln x) + b\sin(\ln x)$ then,

(a) $x^2y_2 + xy_1 + y = 0$

(b) $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.

(vi) If $y = \ln(x + \sqrt{1+x^2})$ then,

(a) $(1+x^2)y_2 + xy_1 = 0$

(b) $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$.

(vii) If $y = e^{m\cos^{-1}x}$ then

(a) $(1-x^2)y_2 - xy_1 = m^2y$.

(b) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} = (m^2+n^2)y_n$.

(viii) If $y = \sin(m\sin^{-1}x)$ then

(a) $(1-x^2)y_2 - xy_1 + m^2y = 0$

(b) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

(ix) If $y = \tan^{-1}x, \quad 0 < x < \dfrac{\pi}{2}$, then

(a) $(1+x^2)y_2 + 2xy_1 = 0$.

(b) $(1+x^2)y_{n+2} + 2x(n+1)y_{n+1} + n(n+1)y_n = 0, \forall n \in \mathbb{N}$.

Also find $y_n(0)$.

(x) If $y = \cos(m\sin^{-1}x)$ then $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

(xi) If $y = (\sin^{-1}x)^2$ then $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$.

xxxxxxxxxxxx

3.5 Practical 1.5: Mean value theorems and applications, L'Hospital's Rule, Increasing and Decreasing functions.

3.5.1 Prerequisite for Practical 1.5

(1) **Rolle's Theorem**: If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) and if, $f(a) = f(b)$, then there is $c \in (a,b)$ such that $f'(c) = 0$.

(2) **Lagrange's Mean Value Theorem**: If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b) then there is $c \in (a,b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{(b-a)}$.

(3) **Cauchy's Mean Value Theorem**: If $f, g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b), then there is $c \in (a,b)$ such that $g'(c)[f(b)-f(a)] = f'(c)[g(b) - g(a)]$.

(4) **Monotonically increasing function**: Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. f is said to be (monotonically) increasing on I if $x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.
Monotonically decreasing function: Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. f is said to be (monotonically) increasing on I if $x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) \geq$

$f(x_2)$.
Monotonic function: Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function.f is said to be monotonic on I if f is monotonically increasing on I or f is monotonically decreasing on I.
Strictly increasing: Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. f is said to be strictly increasing on I if $x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.
Strictly decreasing: Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. f is said to be strictly decreasing on I if $x_1, x_2 \in I, x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

(5) Let I be an interval containing more than one point, and $f : I \longrightarrow \mathbb{R}$ be a differentiable function Then we have the following:
(i) $f' \geq 0$ throughout $I \iff f$ is monotonically increasing on I.
(ii)$f' \leq 0$ throughout $I \iff f$ is monotonically decreasing on I.
(iii)$f' > 0$ throughout $I \implies f$ is strictly increasing on I.
(iv)$f' < 0$ throughout $I \implies f$ is strictly decreasing on I.

(6) **L'Hôpital's Rule**

(i) L'Hôpital's Rule for $\frac{0}{0}$ Indeterminate Form when $x \longrightarrow c, c \in \mathbb{R}$:
Let $c \in \mathbb{R}$ and $D = (c - r, c) \cup (c, c + r)$ for some $r > 0$. Let $f, g : D \longrightarrow \mathbb{R}$ be differentiable functions such that

$$\lim_{x \longrightarrow c} f(x) = 0 \text{ and } \lim_{x \longrightarrow c} g(x) = 0.$$

Suppose $g'(x) \neq 0$ for all $x \in D$, and $\lim_{x \longrightarrow c} \frac{f'(x)}{g'(x)} = l$. Then $\lim_{x \longrightarrow c} \frac{f(x)}{g(x)} = l$. Here l can be a real number or ∞ or $-\infty$.

(ii) L'Hôpital's Rule for $\frac{0}{0}$ Indeterminate Forms when $x \longrightarrow \infty$:
Let $a \in \mathbb{R}$ and let $f, g : (a, \infty) \longrightarrow \mathbb{R}$ be differentiable functions such that

$$\lim_{x \longrightarrow \infty} f(x) = 0 \text{ and } \lim_{x \longrightarrow \infty} g(x) = 0.$$

Suppose $g'(x) \neq 0$ for all $x \in (a, \infty)$, and $\lim_{x \longrightarrow \infty} \frac{f'(x)}{g'(x)} = l$. Then $\lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} = l$. Here l can be a real number or ∞ or $-\infty$.

(iii) L'Hôpital's Rule for $\frac{0}{0}$ Indeterminate Forms when $x \longrightarrow -\infty$:
Let $a \in \mathbb{R}$ and let $f, g : (-\infty, a) \longrightarrow \mathbb{R}$ be differentiable functions such that

$$\lim_{x \longrightarrow -\infty} f(x) = 0 \text{ and } \lim_{x \longrightarrow -\infty} g(x) = 0.$$

Suppose $g'(x) \neq 0$ for all $x \in (-\infty, a)$, and $\lim_{x \longrightarrow -\infty} \frac{f'(x)}{g'(x)} = l$. Then $\lim_{x \longrightarrow -\infty} \frac{f(x)}{g(x)} = l$. Here l can be a real number or ∞ or $-\infty$.

(iv) L'Hôpital's Rule for $\frac{\infty}{\infty}$ Indeterminate Forms when $x \longrightarrow c, c \in \mathbb{R}$:
Let $c \in \mathbb{R}$ and $D = (c - r, c) \cup (c, c + r)$ for some $r > 0$. Let $f, g : D \longrightarrow \mathbb{R}$ be differentiable functions such that

$$|g(x)| \longrightarrow \infty \text{ as } x \longrightarrow c.$$

Suppose $g'(x) \neq 0$ for all $x \in D$, and $\lim_{x \longrightarrow c} \frac{f'(x)}{g'(x)} = l$. Then $\lim_{x \longrightarrow c} \frac{f(x)}{g(x)} = l$. Here l can be a real number or ∞ or $-\infty$.

(v) L'Hôpital's Rule for $\frac{\infty}{\infty}$ Indeterminate Forms when $x \longrightarrow \infty$:
Let $a \in \mathbb{R}$ and let $f, g : (a, \infty) \longrightarrow \mathbb{R}$ be differentiable functions such that

$$|g(x)| \longrightarrow \infty \text{ as } x \longrightarrow \infty.$$

Suppose $g'(x) \neq 0$ for all $x \in (a, \infty)$, and $\lim_{x \longrightarrow \infty} \frac{f'(x)}{g'(x)} = l$. Then $\lim_{x \longrightarrow \infty} \frac{f(x)}{g(x)} = l$.
Here l can be a real number or ∞ or $-\infty$.

(vi) L'Hôpital's Rule for $\frac{\infty}{\infty}$ Indeterminate Forms when $x \longrightarrow -\infty$:
Let $a \in \mathbb{R}$ and let $f, g : (-\infty, a) \longrightarrow \mathbb{R}$ be differentiable functions such that

$$|g(x)| \longrightarrow \infty \text{ as } x \longrightarrow -\infty.$$

Suppose $g'(x) \neq 0$ for all $x \in (-\infty, a)$, and $\lim_{x \longrightarrow -\infty} \frac{f'(x)}{g'(x)} = l$. Then $\lim_{x \longrightarrow -\infty} \frac{f(x)}{g(x)} = l$.
Here l can be a real number or ∞ or $-\infty$.

3.5.2 PRACTICAL 1.5

(A) Objective Questions:

Choose correct alternative in each of the following:

(1) Which of the following functions is increasing on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

(a) $f(x) = x^2$ (b) $f(x) = \cos x$ (c) $f(x) = \sin x$ (d) $f(x) = |x|$

(2) Which of the following functions is decreasing on the interval $(0, \pi)$?

(a) $f(x) = x^2$ (b) $f(x) = \cos x$ (c) $f(x) = \sin x$ (d) $f(x) = \cot x$

(3) The function $y = x^2$ is increasing on

(a) $\mathbb{R}$ (b) $(-\infty, 0)$ (c) $(0, \infty)$ (d) None of these.

(4) The function $y = x^3 - 6x^2 + 9x - 3$ is decreasing on

(a) (1,3) (b) $\mathbb{R}$ (c) $(0,\infty)$ (d) None of these.

(5) The function $y = 5 - 3x^2 + x^3$ is increasing on

(a) $(-\infty, 2) \cup (2, \infty)$ (c) $\mathbb{R}$
(b) $(0, 2)$ (d) None of these.

(6) The function $y = x \ln x$ is decreasing on

(a) $\left(0, \frac{1}{e}\right)$ (b) $\left(\frac{1}{e}, \infty\right)$ (c) $\mathbb{R}$ (d) None of these.

(7) $\lim\limits_{x \to \infty} x \sin \dfrac{1}{x} =$

(a) 0 (b) 1 (c) ∞ (d) None of these.

(8) $\lim\limits_{x \to 0} x^x =$

(a) 1 (b) 0 (c) ∞ (d) None of these.

(9) $\lim\limits_{x \to 0} \left(\dfrac{\ln(1 - x^2)}{\ln(\cos x)} \right) =$

(a) 2 (b) 1 (c) 0 (d) None of these.

(10) $\lim\limits_{x \to a} (x - a)^{x-a} =$

(a) 1 (b) 0 (c) a (d) None of these.

(11) $\lim\limits_{x \longrightarrow o} x \ln \tan x =$

(a) 0 (b) 1 (c) ∞ (d) None of these.

(12) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function such that f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$. Then

(a) there exists a unique $c \in (a, b)$ such that $f'(c) = 0$.

(b) there exists $c \in (a, b)$ such that $f(c) = 0$.

(c) there exists $c \in (a, b)$ such that $f'(c) = 0$.

(d) None of these.

(13) Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function such that f is continuous on $[a, b]$ and differentiable on (a, b). Then

(a) there exists a unique $c \in (a, b)$ such that $f'(c)\frac{f(b)-f(c)}{b-a}$.

(b) there exists $c \in (a, b)$ such that $f(c) = \frac{f(b)-f(a)}{b-a}$.

(c) there exists $c \in (a, b)$ such that $f(c) = \frac{f(b)-f(a)}{b-a}$.

(d) none of these.

(14) Let a and b be such that $0 < a < b < \frac{\pi}{2}$. Then, there exists c between a and b such that

(a) $\frac{\sin a - \sin b}{\cos b - \cos a} = -\cot c$.

(b) $\frac{\sin a - \sin b}{\cos b - \cos a} = \tan c$.

(c) $\frac{\sin a - \sin b}{\cos b - \cos a} = \cot c$.

(d) None of these. exertion

(B) Descriptive Questions

(1) Examine the applicability of Rolle's theorem for the following functions:

(i) $f(x) = x^4, x \in [-2, 2]$

(ii) $f(x) = \sin x + \cos x - 6, x \in [0, 2\pi]$

(iii) $f(x) = \sqrt{1 - x^2}, x \in [-1, 1]$

(iv) $f(x) = \sin x - 1, x \in \left[\frac{pi}{2}, \frac{5\pi}{2}\right]$

(v) $f(x) = (x + 1)(x - 2)^2, x \in [1, 2]$

(vi) $f(x) = x^2 - 5x + 9, x \in [1, 4]$

(vii) $f(x) = \sin 3x, x \in [0, \pi]$

(viii) $f(x) = \cos 2x, x \in [o, \pi]$

(ix) $f(x) = (x - 1)(2x - 3), x \in [1, 3]$

(x) $f(x) = 2 + (x - 1)^{\frac{2}{3}}, x \in [0, 2]$

(xi) $f(x) = |x|, x \in [-1, 1]$

(xii) $f(x) = x^3 - 3x^2 + 2x + 2, x \in [0, 1]$

(2) Consider $f : [0, 1] \longrightarrow \mathbb{R}, f(x) = \begin{cases} x & \text{if } 0 \leq x < 1, \\ 0 & \text{if } x = 1. \end{cases}$ Answer the following questions.

(i) Is f differentiable on $(0, 1)$? If yes, find $f'(x)$ for all $x \in (0, 1)$.

(ii) Is $f(0) = f(1)$?

(iii) Does there exist $c \in (0, 1)$ such that $f'(c) = 0$?

(iv) If your answer to the above question is no, then which condition of Rolle's theorem is not satisfied? Mention the point at which the one of the Rolle's Theorem's condition is not satisfied.

(3) Consider $f : [-1, 1] \longrightarrow \mathbb{R}, f(x) = |x|$.

Answer the following questions.

(i) Is f continuous on $[-1, 1]$?

(ii) Is $f(-1) = f(1)$?

(iii) What is $f'(x)$ for $-1 < x < 0$?

(iv) What is $f'(x)$ for $0 < x < 1$?

(v) Does there exist $c \in (-1, 1)$ such that $f'(c) = 0$?

(vi) If your answer to the above question is no, then which condition of Rolle's theorem is not satisfied? Mention the point at which the one of the Rolle's Theorem's condition is not satisfied.

(4) Consider $f : [0, 1] \longrightarrow \mathbb{R}, f(x) = x$.
Answer the following questions.

(i) Is f continuous on $[0, 1]$?

(ii) Is f differentiable on $(0, 1)$?

(iii) What is $f'(x)$ for $0 < x < 1$?

(iv) Does there exist $c \in (0, 1)$ such that $f'(c) = 0$?

(v) If your answer to the above question is no, then which condition of Rolle's theorem is not satisfied?

(5) Show that the following polynomial functions have exactly one root.

(i) $x^3 + 4x + 1 = 0$

(ii) $x^3 + 4x - 3 = 0$

(iii) $4x^5 + x^3 + 2x + 1$

(iv) $2x - 1 - \sin x = 0$

(6) Show that the equation $x^3 - 15x + c = 0$ has at most one root in the interval $[-2, 2]$.

(7) At what points on the curve $y = \cos x - 1$ on $[0, \pi]$ is the tangent parallel to $x-$ axis.

(8) Examine the applicability of Lagrange's Mean Value Theorem for the following functions.

(i) $f(x) = x(2 - x)$ in $[0, 1]$.

(ii) $f(x) = x(x + 4)^2$ in $[0, 4]$.

(iii) $f(x) = x + \dfrac{1}{x}$ in $[1, 3]$.

(iv) $f(x) = x(x - 2)$ in $[0, 1]$.

(v) $f(x) = \sqrt{x^2 - 4}$ in $[2, 4]$.

(vi) $f(x) = \dfrac{1}{4x - 1}$ in $[1, 4]$.

(vii) $f(x) = x^2 - 3x - 1$ in $[- \left[\frac{11}{7}, \frac{13}{7}\right]$.

(viii) $f(x) = e^x$ in $[0, 1]$.

(ix) $f(x) = x(x + 4)^2$ in $[0, 4]$.

(x) $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$.

(9) Use LMVT to prove the following.

(i) $|\sin x - \sin y| \le |x - y|$, for all $x, y \in \mathbb{R}$.

(ii) $|\cos x - \cos y| \le |x - y|$, for all $x, y \in \mathbb{R}$.

(10) State the intervals on which f is increasing or decreasing.

(i) $f(x) = 5 - 3x^2 + x^3$

(ii) $f(x) = 3 - 2x^2 + x^4$

(iii) $f(x) = 91 + 9x - 6x^2 + x^3$

(iv) $f(x) = 71 + 18x - 12x^2 + 2x^3$

(v) $f(x) = x - 2\sin x, 0 < x < 3\pi$

(vi) $f(x) = \cos^2 x - 2\sin x, 0 \le x \le 2\pi$

(vii) $f(x) = xe^x$

(viii) $f(x) = x^4 + 6x^3 + 17x^2 + 32x + 32$

(11) Show that the following functions are either increasing or decreasing for all real values of x.

(i) $f(x) = 10 - 12x - 3x^2 - x^3$

(ii) $f(x) = x^3 - 9x^2 + 30x + 13$

(iii) $f(x) = 4 - e^{5x}$

(iv) $f(x) = \dfrac{1 - 2x - x^2}{1 + x - 2x^2}$

(12) Show that $\tan x > x$ if $0 < x < \frac{\pi}{2}$

(13) Show that $-\dfrac{x^2}{2} < \ln(1 + x) < x - \dfrac{x^2}{2(1 + x)}$, for each $x > 0$.

(14) Show that $1+x<e^x<1+xe^x$, for all $x\geq 0$.

(15) Show that $\dfrac{x}{1+x}<\ln(1+x)<x$, for all $x>0$, hence show that $0<\dfrac{1}{\ln(1+x)}-\dfrac{1}{x}<1$, for all $x>0$.

(16) Show that $\dfrac{\tan x}{x}>\dfrac{x}{\sin x}$, for $0<x<\dfrac{\pi}{2}$.

(17) Show that $x^2-1>2x\ln x>4(x-1)-2\ln x$, for all $x\in(1,\infty)$.

(18) Find c of Cauchy's Mean Value Theorem.

(i) $f(x)=x(x-2)(x-3)$ and $g(x)=x(x-1)(x-2), x\in\left[0,\dfrac{1}{2}\right]$.

(ii) $f(x)=\cos x, g(x)=\sin x, x\in\left[-\dfrac{\pi}{2},0\right]$.

(19) Using L'Hôpital's Rule, evaluate the following limits.

(i) $\displaystyle\lim_{x\longrightarrow 0}\frac{x^2+2\cos x-2}{x\sin^3 x}$

(ii) $\displaystyle\lim_{x\longrightarrow 0}\frac{1+\sin x-\cos x+\ln(1-x)}{x\tan^2 x}$

(iii) $\displaystyle\lim_{x\longrightarrow 0}\frac{\ln\sin x}{\cot x}$

(iv) $\displaystyle\lim_{x\longrightarrow\frac{\pi}{2}}\frac{\tan 2x}{\tan x}$

(v) $\displaystyle\lim_{x\longrightarrow 0}x\ln x$

(vi) $\displaystyle\lim_{x\longrightarrow 0}\frac{1}{x^2}-\frac{1}{\sin^2 x}$

(vii) $\displaystyle\lim_{x\longrightarrow 0}x^x$

(viii) $\displaystyle\lim_{x\longrightarrow\frac{\pi}{2}}(\sec x)^{\cot x}$

(ix) $\displaystyle\lim_{x\longrightarrow 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x^2}}$

(x) $\displaystyle\lim_{x\longrightarrow 1}x^{1-x}$

(20) Let f,g and h be continuous on $[a,b]$ and differentiable on (a,b), then show that there exists a point $c\in(a,b)$ such that $\begin{vmatrix} f(a) & f(b) & f'(c)\\ g(a) & g(b) & g'(c)\\ h(a) & h(b) & h'(c)\end{vmatrix}=0$. (Hint: Consider the function $k:[a,b]\longrightarrow\mathbb{R}$ as $k(t)=\begin{vmatrix} f(a) & f(b) & f(t)\\ g(a) & g(b) & g(t)\\ h(a) & h(b) & h(t)\end{vmatrix}$. Check if you can apply LMVT and proceed accordingly.)

(21) If $f(1)=10$ and $f'(x)\geq 2$ for $1\leq x\leq 4$. Determine how small can $f(4)$ become?

(22) Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function. If f' is continuous, use L'Hôpital's Rule to show that $\displaystyle\lim_{h\longrightarrow o}\frac{f(x+h)-f(x-h)}{2h}=f'(x)$.

(23) Evaluate the following limits.

(i) $\displaystyle\lim_{x\to 1}\frac{1+\log x-x}{1-2x+x^2}$

(ii) $\displaystyle\lim_{x\to 0}\frac{\sin hx-x}{\sin x-x\cos x}$

(iii) $\displaystyle\lim_{x\to 0}\frac{\log(1-x^2)}{\log\cos x}$

(iv) $\displaystyle\lim_{x\to 0}\frac{xe^x-\log(1+x)}{x^2}$

3.5

(v) $\lim_{x\to 0} \frac{e^x \sin x - x - x^2}{x^2 + x\log(1-x)}$

(vi) $\lim_{x\to 0} \frac{\sin x - \log(e^x \cos x)}{x \sin x}$

(vii) $\lim_{x\to 0} \log(1-x) \cot \frac{\pi x}{2}$

(viii) $\lim_{x\to 0} \frac{\tan^2 x - x^2}{x^2 \tan^2 x}$

(ix) $\lim_{x\to \pi/2} \frac{\tan x}{\tan 3x}$

(x) $\lim_{x\to 0} x \tan(\frac{\pi}{2} - x)$

(xi) $\lim_{x\to 0} (\cot x) \sin 2x$

(xii) $\lim_{x\to 0} (\frac{2x+1}{x+1})^{x-1}$

(xiii) $\lim_{x\to 0} (\frac{\sin hx}{x})^{x-2}$

(xiv) $\lim_{x\to \pi/4} (\tan x)^{\tan 2x}$

(xv) $\lim_{x\to 0+} (\cot x)^{\sin x}$

(xvi) $\lim_{x\to \frac{1}{2}} \frac{\cos^2 \pi x}{e^2 x - 2e^x}$

(xvii) $\lim_{x\to 0} \frac{xe^x - \log(1+x)}{\cos hx - \cos x}$

(xviii) $\lim_{x\to 0} \frac{a^{ax} - e^{-ax}}{\log(1+bx)}$

(xix) $\lim_{x\to 0} \frac{x\cos x - \log(1+x)}{x^2}$

(xx) $\lim_{x\to 0} \frac{x^2 + 2\cos x - 2}{x \sin^3 x}$

(xxi) $\lim_{x\to 0} \frac{x - \log(1+x)}{1 - \cos x}$

(xxii) $\lim_{x\to 0} \frac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$

(xxiii) $\lim_{x\to 0} \frac{\log \sin x}{\cot x}$

(xxiv) $\lim_{x\to 0} \log_{\tan x} \tan 2x$

(xxv) $\lim_{x\to 0} (\frac{1}{x^2} - \cot^2 x)$

(xxvi) $\lim_{x\to \pi/2} (\sin x)^{\tan x}$

(xxvii) $\lim_{x\to 0} (1 + \sin x)^{\cot x}$

(xxviii) $\lim_{x\to 0} \frac{1}{x^x - 1}$

(xxix) $\lim_{x\to \pi/2+} (\sec x)^{\cos x}$

xxxxxxxxxxx

3.6 Practical 1.6: Extreme values, Taylor's Theorem and Curve Sketching.

3.6.1 Prerequisite for Practical 1.5

(1) **Taylor's Theorem: (Lagranges form of remainder)** Let $n \in \mathbf{Z}, n \geq 0$, and $f : [a, b] \to \mathbb{R}$ be such that $f', f'', \cdots, f^{(n)}$ exist on $[a, b]$and further, $f^{(n)}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there is $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \tfrac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{(n+1)}.$$

(2) Let $a \in \mathbb{R}$ and I be an interval in $\mathbb{R}$ such that a is an interior point of I. Suppose $f : I \longrightarrow \mathbb{R}$ is an infinitely differentiable function. The series around the point X given by $\sum_{k\geq 0} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is called the **Taylor series** of f around a. In the special case that $a = 0$, the Taylor series of f around a is sometimes called the **Maclaurin series** of f.

(3) **Taylor's Polynomial of f around a**: The polynomial given by
$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the n^{th} **Taylor polynomial** of f around a.

(4) **Convex**: Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. f is **convex** on I or **concave upward on I** if $x_1, x_2, x \in I, x_1 < x < x_2 \Rightarrow f(x) - f(x_1) \leq \frac{f(x_2)-f(x_1)}{x_2-x_1}(x - x_1)$
Therefore f is convex on I if $x_1, x_2, x \in I, x_1 < x < x_2 \Rightarrow f(x) \leq f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(x - x_1)$.
Now if we denote $f(x_1) + \frac{f(x_2)-f(x_1)}{x_2-x_1}(x - x_1)$ by $L(x)$ then f is convex on I if and only if $x_1, x_2, x \in I, x_1 < x < x_2 \Rightarrow f(x) \leq L(x)$ i.e. f is convex on I if and only of the graph of f lies below the line joining any two points on it since $L(x)$ is nothing but the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

(5) **Concave**: Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. f is **concave** on I or **concave downward on I** if $x_1, x_2, x \in I, x_1 < x < x_2 \Rightarrow f(x) - f(x_1) \geq \frac{f(x_2)-f(x_1)}{x_2-x_1}(x - x_1)$.
As above we can say that f is concave on I if and only if $x_1, x_2, x \in I, x_1 < x < x_2 \Rightarrow f(x) \geq L(x)$
i.e. f is convex on I if and only of the graph of f lies above the line joining any two points on it.

(6) **Critical point:** Let $D \subseteq \mathbb{R}$ and $f : D \longrightarrow \mathbb{R}$, a point $c \in D$ is called a **critical point** of f if c is an interior point of D such that either f is not differentiable at c, or f is differentiable at c and $f'(c) = 0$.

(7) **Point of Inflection**: Let I be an interval in $\mathbb{R}$ and $f : I \longrightarrow \mathbb{R}$ be a function. Let c be an interior point of I then c is a point of inflection for f if there is $\delta > 0$ such that f is convex in $(c - \delta, c)$, while f is concave in $(c, c + \delta)$, or vice versa, that is, f is concave in $(c - \delta, c)$, while f is convex in $(c, c + \delta)$.

(8) **Necessary Condition for a Point of Inflection:** Let $D \subseteq \mathbb{R}$, c be an interior point of D, and $f : D \to \mathbb{R}$ be a function. Let f be twice differentiable at c. If c is a point of inflection for f, then $f''(c) = 0$.

(9) **Sufficient condition for a Point of Inflection:** Let $D \subseteq \mathbb{R}$, c be an interior point of D, and $f : D \to \mathbb{R}$ be a function such that f is thrice differentiable at c. If $f''(c) = 0$ and $f^3(c) \neq 0$, then c is a point of inflection for f.

(10) Let I be an interval containing more than one point, and $f : I \to \mathbb{R}$ be a twice differentiable function. Then we have the following
(i) $f'' \geq 0$ throughout $I \quad \longleftrightarrow \quad f$ is convex on I
(ii) $f'' \leq 0$ throughout $I \quad \longleftrightarrow \quad f$ is concave on I
(iii) $f'' > 0$ throughout $I \quad \to \quad f$ is strictly convex on I
(iv) $f'' < 0$ throughout $I \quad \to \quad f$ is strictly concave on I.

(11) **Local Maximum**: If $D \subseteq \mathbb{R}$ and c is an interior point in D, then $f : D \longrightarrow \mathbb{R}$ is said to have a local maximum at c if there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq D$ and $f(x) \leq f(c)$ for all $x \in (c - \delta, c + \delta)$.

3.6

(12) **Local Minimum:** If $D \subseteq \mathbb{R}$ and c is an interior point in D, then $f : D \longrightarrow \mathbb{R}$ is said to have a local minimum at c if there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq D$ and $f(x) \geq f(c)$ for all $x \in (c - \delta, c + \delta)$.

(13) **Strict Local Maximum**: If $D \subseteq \mathbb{R}$ and c is an interior point in D, then $f : D \longrightarrow \mathbb{R}$ is said to have a strict local maximum at c if there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq D$ and $f(x) < f(c)$ for all $x \in (c - \delta, c + \delta)$.

(14) **Strict Local Minimum**: If $D \subseteq \mathbb{R}$ and c is an interior point in D, then $f : D \longrightarrow \mathbb{R}$ is said to have a strict local minimum at c if there is $\delta > 0$ such that $(c - \delta, c + \delta) \subseteq D$ and $f(x) > f(c)$ for all $x \in (c - \delta, c + \delta)$.

(15) **Necessary condition for f to have a local extremum at an interior point c of $D \subseteq \mathbb{R}$:**
Let $D \subseteq \mathbb{R}$ and c be an interior point of D. If $f : D \longrightarrow \mathbb{R}$ is differentiable at c and has a local extermum at c, then $f'(c) = 0$.

(16) **First Derivative Test for Local Minimum** Let $D \subseteq \mathbb{R}, c$ be an interior point of D, and $f : D \longrightarrow \mathbb{R}$ be any function. If :

(i) f is continuous at c, and also,

(ii) f is differentiable on $(c - r, c) \bigcup (c, c + r)$ for some $r > 0$, and

(iii) there is $\delta > 0$ with $\delta \leq r$ such that $f'(x) \leq 0$ for all $x \in (c - \delta, c)$ and $f'(x) \geq 0$ for all $x \in (c, c + \delta)$

then f has a local minimum at c.

(17) **Second Derivative Test for Local Minimum**: If $f : (a, b) \longrightarrow \mathbb{R}$ is twice differentiable at $c \in (a, b)$ and satisfies $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(18) **First Derivative Test for Local Maximum** Let $D \subseteq \mathbb{R}, c$ be an interior point of D, and $f : D \longrightarrow \mathbb{R}$ be any function. If:

(i) f is continuous at c, and also,

(ii) f is differentiable on $(c - r, c) \bigcup (c, c + r)$ for some $r > 0$, and

(iii) there is $\delta > 0$ with $\delta \leq r$ such that $f'(x) \geq 0 \ \forall x \in (c - \delta, c)$ and $f'(x) \leq 0 \ \forall x \in (c, c + \delta)$

then f has a local maximum at c.

(19) **Second Derivative Test for Local Maximum**: If $f : (a, b) \longrightarrow \mathbb{R}$ is twice differentiable at $c \in (a, b)$ and satisfies $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

(A) Objective Questions

Choose correct alternative in each of the following:

(1) $f(x) = x^{\frac{4}{3}} - 4x^{\frac{1}{3}}$. Then has critical points at

(a) $x = 1$ is the only critical point of f.
(b) $x = 0$ is the only critical point of f.
(c) $x = 0, 1$ both are critical points of f.
(d) None of these.

(2) The function $y = \frac{x^2+1}{x}$ attains its maximum value at

(a) 1 (b) -1 (c) 0 (d) None of these

(3) The function $y = \sin x, x \in \mathbb{R}$ attains its minimum value

(a) at exactly one point
(b) at only finitely many points
(c) at infinitely many points
(d) nowhere

(4) For a given function, $y = f(x)$, it is found that $f'(c) = 0$. Therefore,

(a) c must be a point of local maximum of f.
(b) c must be a point of local minimum of f.
(c) c must be a point of either local maximum or local minimum of f.
(d) nothing can be said about c.

(5) Amongst the following, the function having a local minimum at the origin, is

(a) $y = \sin x$
(b) $y = x^3$
(c) $y = |x|$
(d) $y = x^2 - 2x + 1$

(6) Amongst the following, the function having a local maximum at the origin, is

(a) $y = \cos x$
(b) $y = x^2$
(c) $y = |x|$
(d) $y = x^2 + 2x + 1$

(7) The function $y = \sin x + \cos x, x \in (\frac{-\pi}{2}, \frac{\pi}{2})$ has local maximum at

(a) $\frac{\pi}{4}$
(b) $\frac{-\pi}{4}$
(c) 0
(d) None of these

(8) A rectangle with the given perimeter has maximum area if and only if it's a

(a) rhombus
(b) parallelogram
(c) square
(d) kite

(9) A triangle with the given perimeter has maximum area if and only if it is

(a) obtuse angled
(b) isosceles
(c) right angled
(d) equilateral

(10) The function $y = e^x$ is concave upwards

(a) only at the origin
(b) over negative real numbers only.
(c) over positive real numbers only
(d) everywhere .

(11) Consider the polynomial function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^4 + 2x^3 - 36x^2 + 62x + 5$. Then f is

(a) convex on $(-\infty, -3] \cup [2, \infty)$ and concave on $[-3, 2]$.
(b) convex on $(-\infty, -2] \cup [3, \infty)$ and concave on $[-2, 3]$.
(c) convex on $[-2, \infty)$ and concave on $(-\infty, -3]$.
(d) none of these.

(12) Let $n \in \mathbb{N}$ and consider the nth power function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = x^n$. Which of the following statement is Not True?

(a) If n is even then f is convex on $\mathbb{R}$.
(b) If n is odd and $n > 1$ then f is convex on $[0, \infty)$ and concave on $(-\infty, 0]$ and concave on $[-2, 3]$.
(c) convex on $[-2, \infty)$ and concave on $(-\infty, -3]$.
(d) none of these.

(13) The function $y = \ln x$ is concave downwards

(a) only at 1
(b) over positive rational numbers only.
(c) over positive irrational numbers only.
(d) wherever it is defined.

(14) The n^{th} Taylor polynomial of $\sin x$ around 0 is given by

(a) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{(k-1)} x^{2k-1}}{(2k-1)!}$ where $n = 2k$ or $n = 2k - 1$.
(b) $1 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots + \frac{(-1)^{(k)} x^{2k-1}}{(2k-1)!}$ where $n = 2k$ or $n = 2k - 1$.
(c) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!}$ where $n = 2k$ or $n = 2k - 1$.
(d) none of these.

(15) The n^{th} Taylor polynomial of $\cos x$ around 0 is given by

(a) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{(k-1)} x^{2k-1}}{(2k-1)!}$ where $n = 2k$ or $n = 2k - 1$.
(b) $1 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \cdots + \frac{(-1)^{(k)} x^{2k-1}}{(2k-1)!}$ where $n = 2k$ or $n = 2k - 1$.
(c) $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{(-1)^k x^{2k}}{(2k)!}$ where $n = 2k$ or $n = 2k + 1$.
(d) none of these.

(16) The n^{th} Taylor polynomial of $f(x) = \frac{1}{1-x}$ around 0 is

(a) $1 + x + x^2 + \cdots + x^n$.

(b) $1 - x + x^2 + \cdots + (-1)^n * x^n$.

(c) $1 - x + x^2/2 + \cdots + (-1)^n * x^n/n$.

(d) none of these.

(17) The n^{th} Taylor polynomial of e^x around 0 is given by

(a) $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{(-1)^{(n-1)} x^{2n-1}}{(2n-1)!}$.

(b) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}$.

(c) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{x^{2n}}{(2n)!}$.

(d) none of these.

(B) Descriptive Questions

(1) Find the points of local extrema for the following functions.

(i) $f(x) = 5x^6 + 18x^5 + 15x^4 - 10$

(ii) $f(x) = \dfrac{(x+1)(x+4)}{(x-1)(x-4)}$ for $x \neq 1, 4$.

(iii) $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 1$

(iv) $f(x) = \dfrac{x^3}{x^4 + 1}$

(v) $f(x) = (x-1)(x-2)(x-3)$

(2) Find the local extreme values of the following functions.

(i) $f(x) = \sin^2 x(1 + \cos x)^3$

(ii) $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 1$

(iii) $f(x) = \dfrac{x^3}{x^4 + 1}$

(iv) $f(x) = (x-1)(x-2)(x-3) - 24x$

(v) $f(x) = \dfrac{\ln x}{x}$

(vi) $f(x) = x \ln x$

(vii) $f(x) = x^x$

(viii) $f(x) = xe^{-x}$

(3) Show that the greatest value of $x^m(a-x)^n$ is $\dfrac{m^m n^n a^{m+n}}{(m+n)^{m+n}}$

(4) Divide 100 into the parts such that sum of their squares is minimum.

(5) Show that among all the rectangles with a fixed area, square has the least perimeter.

(6) Show that among all the rectangles with a fixed perimeter , square has the maximum area.

(7) Find the maximum area of a triangle, given that its perimeter is 24 units.

(8) Find the points of inflection.

(i) $y = x^4 - 6x^2 + 8x - 1$

(ii) $y^2 = x(x+1)^2$

(iii) $y = \dfrac{x^3}{a^2 + x^2}$

(iv) $y = x^3 - 9x^2 + 7x - 6$

(9) Show that $y = e^x$ is concave upwards (convex) for all values of x.

(10) Show that $y = \ln x$ is concave downwards (concave) for all values of x.

(11) Prove that the points of inflections of the following curves li on a straight line.

3.6

(i) $y^2 = (x-a)^2 - (x-b)$ (ii) $x^2y + a^2(x+y) = a^3$

(12) Examine the curve $y = \cos x$ for concavity in the interval $[-2\pi, 2\pi]$. Also indicate the points of inflection.

(13) Find the local maximum and minimum values of f. Also find the intervals of concavity and the inflection points.

(i) $f(x) = x^4 - 2x^2 + 3$
(ii) $f(x) = xe^x$
(iii) $f(x) = 2x^3 - 3x^2 - 12x$
(iv) $f(x) = 2\cos x - \cos 2x, 0 \le x \le 2\pi$.
(v) $f(x) = \sqrt{x^2+1} - x$
(vi) $f(x) = x - 2\sin x, 0 < x < 3\pi$.
(vii) $f(x) = \dfrac{\ln x}{\sqrt{x}}$
(viii) $f(x) = x^4 - 6x^2$

(14) Show that the function $f(x) = \sin x, x \in (-\pi, \pi)$ has extrema at $\pi\dfrac{\pi}{2}$.

(15) Find the local extremum of $f(x) = x^x$.

(16) Sketch the following curves.

(i) $y = x^2(x-1)$ (ii) $y = x^3 - x$ (iii) $|x+y| = 2$

(17) write n^{th} Taylor polynomial of $\cosh x$ and $\sinh x$ around 0. (Hint: $\cosh x = \dfrac{e^x + e^{-x}}{2}, \sinh x = \dfrac{e^x - e^{-x}}{2}$)

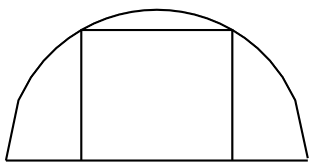
(18) Expand $2 + x^2 - 3x^5 + 7x^6$ in powers of $(x-1)$.

(19) Use Taylor's theorem to find $\sqrt[4]{80.7}$ correct upto 4 decimal places.

(20) Use Taylor's theorem to find $\ln 1.2$ correct upto 4 decimal places.

(21) Use Taylor's theorem to find $\sin 90^0$ correct upto 5 decimal places.

(22) A semicircle of radius 2 units is drawn. A rectangle with base on the diameter of the semicircle is drawn such that the other two vertices of the rectangle are on the semicircle. Find the maximum area of such a rectangle. (Observe the sketch for better understanding of the problem)



(23) Trace the curve of the function $f : [0, \infty) \longrightarrow \mathbb{R}, f(x) = x^3 - 6x^2 + 9x + 1$.

(24) Make a rough sketch of the graph of the function $f(x) = x^3 - 3x^2 + 3$.

(25) Sketch the curve $y = 3x^2 - 2x^3$ for $x \in [-1.5, 2]$.

(26) Trace the curve $f : [0, 2] \to \mathbb{R}$ defined by $f(x) = 1 + (x - 1)^3$.

(27) Trace the curve $f : [-3, 3] \to \mathbb{R}$ defined by $f(x) = -x^3 + 12x + 5$.

xxxxxxxxxxxxxxxxx

3.7 Practical 1.7: Miscellaneous theory questions

3.7.1 Practical 1.7: Miscellaneous theory questions from unit I

(1) Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D point of D. Also, let $f : D \longrightarrow \mathbb{R}$ be a function. Define limit of f as x tends to c. Prove that if $\lim_{x \longrightarrow c} f(x)$ exists then it is unique.

(2) Let $D \subseteq \mathbb{R}, c \in \mathbb{R}$ be a limit point of D and let $f : D \longrightarrow \mathbb{R}$ be such that $\lim_{x \to c} f(x) = L$ for some $L \in \mathbb{R}$. Then prove that

(i) If $L > 0$, then there is $\delta > 0$ such that $f(x) > 0$ for all $x \in D$ satisfying $0 < |x-c| < \delta$.

(ii) If $L < 0$, then there is $\delta > 0$ such that $f(x) < 0$ for all $x \in D$ satisfying $0 < |x-c| < \delta$.

(iii) Hence if $L \neq 0$ then there is $\delta > 0$ such that $f(x) \neq 0$ for all $x \in D$ satisfying $0 < |x - c| < \delta$.

(3) Let $D \subseteq \mathbb{R}, c \in \mathbb{R}$ be a limit point of D and let $f : D \longrightarrow \mathbb{R}$ be such that $\lim_{x \to c} f(x) = L$ for some $L \in \mathbb{R}$. Then prove that there exist $K, \delta > 0$ such that

$$|f(x)| \leq K \text{ for all } x \in D \text{ satisfying } 0 < |x - c| < \delta.$$

(4) Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. If $f : D \longrightarrow \mathbb{R}$ and $\lim_{x \to c} f(x) = L$ for some $L \in \mathbb{R}$. If there is $\delta_1 > 0$, such that

(i) $f(x) > 0$ for all $x \in D$ satisfying $0 < |x - c| < \delta_1$ then prove that $L \geq 0$.

(ii) $f(x) < 0$ for all $x \in D$ satisfying $0 < |x - c| < \delta_1$ then prove that $L \leq 0$.

(5) Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be a limit point of D. If $f, g : D \longrightarrow \mathbb{R}$ and $L, M \in \mathbb{R}$. Then prove that

(i) if $\lim_{x \to c} f(x) = 0$ and there exists $K, \delta_1 > 0$ such that $|g(x)| \leq K$ for all $x \in D$ satisfying $0 < |x - c| < \delta_1$ then $\lim_{x \to c} f(x)g(x) = 0$.

(ii) if $f(x) \leq g(x)$ for all $x \in \mathbb{D}$ and $\lim_{x \longrightarrow p} f(x) = L$, $\lim_{x \longrightarrow p} g(x) = M$ then $L \leq M$.

(iii) If $\lim_{x \to p} f(x) = L$ then $\lim_{x \to p} |f(x)| = |L|$. (Converse not true)

(6) State and prove Sandwich Theorem for limit of a function.

(7) Let $D \subseteq \mathbb{R}, c \in \mathbb{R}$ be a limit point of $D \cap (-\infty, c)$ and let $f : D \longrightarrow \mathbb{R}$ be a function. Define left hand limit of f as x tends to c from the left of c.

(8) Let $D \subseteq \mathbb{R}, c \in \mathbb{R}$ be a limit point of $D \cap (c, \infty)$ and let $f : D \longrightarrow \mathbb{R}$ be a function. Define right hand limit of f as x tends to c from the right of c.

(9) Let $D \subseteq \mathbb{R}$ and let $c \in \mathbb{R}$ be such that c is a limit of $D \cap (-\infty, c)$ as well as of $D \cap (c, \infty)$. Also, let $f : D \longrightarrow \mathbb{R}$ be a function. Then prove that $\lim_{x \longrightarrow c} f(x)$ exists $\Longleftrightarrow$and i $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ exist and are equal.
In this case, $\lim_{x \longrightarrow c} f(x) = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x)$.

(10) Let $D \subseteq \mathbb{R}$, and let $c \in \mathbb{R}$ be a limit point of D. Also, let $f : D \longrightarrow \mathbb{R}$ be a function. Then f is continuous at c if and only if $\lim_{x \longrightarrow c} f(x)$ exists and is equal to $f(c)$.

(11) **Algebra of Continuous functions**: Let $D \subseteq \mathbb{R}, c \in D$ and $f, g : D \longrightarrow \mathbb{R}$ be functions such that f, g are continuous at c. Then prove that

   (i) $f + g$ is continuous at c.

   (ii) rf is continuous at c for every $r \in \mathbb{R}$.

   (iii) fg is continuous at c.

   (iv) If $g(x) \neq 0$ for all $x \in D$ then $\frac{f}{g} : D \longrightarrow \mathbb{R}$ is continuous at c.

(12) Let $D \subseteq \mathbb{R}, c \in D$ and let $f : D \to \mathbb{R}$ be a function. Then prove that f is continuous at c if and only if for each sequence (x_n) in D converging to c, the sequence $(f(x_n))$ converges to $f(c)$.

(13) Let $D, E \subseteq \mathbb{R}$, and let $f : D \longrightarrow \mathbb{R}$ and $g : E \longrightarrow \mathbb{R}$ be functions such that $f(D) \subseteq E$. Let $c \in D$ be such that f is continuous at c and g is continuous at $f(c)$. Then prove that the composite function $g \circ f : D \longrightarrow \mathbb{R}$ is continuous at c.

(14) Let $D \subseteq \mathbb{R}, c \in D, f : D \longrightarrow \mathbb{R}$ be a function that is continuous at c.. Then prove the following.

   (i) f is bounded in some δ neighbourhood of c. That is, there exists $\delta, K > 0$ such that $|f(x)| \leq K$, for all $x \in D$ satisfying $|x - c| < \delta$.

   (ii) $|f| : D \longrightarrow \mathbb{R}$ defined by $(|f|)(x) = |f(x)|$ is also continuous at c. (Converse not true.)

   (iii) If $f(c) > 0$, then there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in D$ satisfying $|x - c| < \delta$.

   (iv) If $f(c) < 0$, then there exists $\delta > 0$ such that $f(x) < 0$ for all $x \in D$ satisfying $|x - c| < \delta$.

(15) State and prove intermediate value theorem for a continuous real valued function defined on $\mathbb{R}$. OR
Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) \neq f(b)$. For any real number k between $f(a)$ and $f(b)$, prove that there exist $c \in (a, b)$ such that $f(c) = k$.

(16) State Bolzano-Weierstrass theorem: If a function $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded and attains its bounds.

3.7.2 Practical 1.7: Miscellaneous theory questions from unit II

(1) Let $D \subseteq \mathbb{R}, c \in D$ be an interior point of D. Let $f : D \longrightarrow \mathbb{R}$ be a function. Define differentiability of f at c. Also prove that if f is differentiable at c then f is continuous at c. Is the converse true? Justify your answer.

(2) (i) Let $D \subseteq \mathbb{R}$ and $c \in D$ be such that $(c-r, c] \subseteq D$ for some $r > 0$. Define the left hand derivative of f at c.

(ii) Let $D \subseteq \mathbb{R}$ and $c \in D$ be such that $[c, c+r) \subseteq D$ for some $r > 0$. Define the right hand derivative of f at c.

(iii) If $f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) = |x|$ for all $x \in \mathbb{R}$, find $f'_-(0)$ and $f'_+(0)$. Hence determine whether f is differentiable at 0 or not. Justify your answer.

Let $D \subseteq \mathbb{R}$, let c be an interior point of D and $f, g : D \longrightarrow \mathbb{R}$ be differentiable at c. Then

(i) $f + g$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$,

(ii) rf is differentiable at c and $(rf)'(c) = rf'(c)$ for every $r \in \mathbb{R}$.

(iii) fg is differentiable at c and $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$.

(iv) If $f(c) \neq 0$ then the function $\dfrac{1}{f}$ is differentiable at c and $\left(\dfrac{1}{f}\right)'(c) = -\dfrac{f'(c)}{(f(c))^2}$

(v) If $g(c) \neq 0$ then the function $\dfrac{f}{g}$ is differentiable at c and $\left(\dfrac{f}{g}\right)'(c) = -\dfrac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}$.

(3) State and prove the Chain Rule.

OR

Let $D, E \subseteq \mathbb{R}$ and $f : D \longrightarrow \mathbb{R}, g : E \longrightarrow \mathbb{R}$ be functions such $f(D) \subseteq E$. Suppose $c \in D$ is an interior point of D such that $f(c)$ is an interior point of E. If f is differentiable at c and g is differentiable at $f(c)$, then prove that the composite function $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'\Big(f(c)\Big) \cdot f'(c)$.

(4) State and prove inverse function theorem.

OR

Let $I \subseteq \mathbb{R}, I$ be an interval and let $c \in I$ be an interior point of I. Suppose $f : I \longrightarrow \mathbb{R}$ is a one-one and continuous function. Let $f^{-1} : f(I) \longrightarrow I$ be the inverse function. If f is differentiable at c and $f'(c) \neq 0$ then f^{-1} is differentiable at $f(c)$ and

$$(f^{-1})'(f(c)) = \frac{1}{f'(c)}.$$

(5) Let $n \in \mathbb{N}$. n^{th} order derivatives of standard functions (y_n denote n^{th} ordered derivative of y)
$a, b \in \mathbb{R}, m, n \in \mathbb{N}$

(i) If $y = (ax+b)^m$, then
$$y_n = \begin{cases} {}^mP_n a^n (ax+b)^{m-n} & \forall\, n \leq m \\ 0 & \forall\, n > m \end{cases}.$$

(ii) If $y = \dfrac{1}{ax+b}, x \neq \dfrac{-b}{a}$ then
$$y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}} \quad \forall\, n \in \mathbb{N}.$$

(iii) If $y = \ln(ax+b), ax+b > 0$ then
$$y_n = \frac{(-1)^{n-1}(n-1)! a^n}{(ax+b)^n} \quad \forall\, n \in \mathbb{N}.$$

(iv) If $y = \sin(ax+b)$, then

$y_n = a^n \sin\left(ax + b + n \cdot \frac{\pi}{2}\right) \quad \forall\, n \in \mathbb{N}.$

(v) If $y = \cos(ax + b)$ then
$y_n = a^n \cos(ax + b + n \cdot \frac{\pi}{2})\ \forall\, n \in \mathbb{N}.$

(vi) If $y = e^{mx}$, then $y_n = m^n e^{mx} \quad \forall\, n \in \mathbb{N}.$

(vii) If $y = a^{mx}, a > 0$ then
$y_n = m^n a^{mx} (\ln a)^n, \quad \forall\, n \in \mathbb{N}.$

(6) State and prove Leibniz Rule of the n^{th} order derivative of the product of two n times differentiable functions, for $n \in \mathbb{N}$.

3.7.3 Practical 1.7: Miscellaneous theory questions from unit III

(1) State and prove Rolle's Theorem.

OR

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) and if, $f(a) = f(b)$, then prove that there is $c \in (a, b)$ such that $f'(c) = 0$.

(2) State and prove Lagrange's Mean Value Theorem.

OR

If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) then prove that there is $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{(b - a)}$.

(3) State and prove Cauchy's Mean Value Theorem.

OR

If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then prove that there is $c \in (a, b)$ such that

$$g'(c)[f(b) - f(a)] = f'(c)[g(b) - g(a)].$$

(4) State and prove Taylor Theorem with Lagrange's form of remainder.

OR

Let $n \in \mathbf{Z}, n \geq 0$, and $f : [a, b] \to \mathbb{R}$ be such that $f', f'', \cdots, f^{(n)}$ exist on $[a, b]$ and further, $f^{(n)}$ is continuous on $[a, b]$ and differentiable on (a, b). Then there is $c \in (a, b)$ such that
$f(b) = f(a) + f'(a)(b - a) + \frac{f''(a)}{2!}(b - a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(b - a)^n + \dfrac{f^{(n+1)}(c)}{(n + 1)!}(b - a)^{(n+1)}.$

(5) Let I be an interval containing more than one point, and $f : I \longrightarrow \mathbb{R}$ be a differentiable function Then prove the following:
(i) $f'(x) \geq 0$ for all $x \in I \iff f$ is monotonically increasing on I.
(ii) $f'(x) \leq 0$ for all $x \in I \iff f$ is monotonically decreasing on I.

(6) Let I be an interval containing more than one point, and $f : I \longrightarrow \mathbb{R}$ be a differentiable function. Then we have the following:

(i) f' is monotonic increasing on $I \Longleftrightarrow f$ is convex on I.

(ii) f' is monotonic decreasing on $I \Longleftrightarrow f$ is concave on I.

(7) Let I be an interval containing more than one point, and $f : I \to \mathbb{R}$ be a twice differentiable function. Then we have the following
(i) $f''(x) \geq 0$ for all $x \in I \quad \longleftrightarrow \quad f$ is convex on I.
(ii) $f''(x) \leq 0$ for all $x \in I \quad \longleftrightarrow \quad f$ is concave on I.

(8) Let $D \subseteq \mathbb{R}$ and c be an interior point of D. If $f : D \longrightarrow \mathbb{R}$ is differentiable at c and has a local extermum at c, then prove that $f'(c) = 0$.

(9) Let $D \subseteq \mathbb{R}, c$ be an interior point of D, and $f : D \longrightarrow \mathbb{R}$ be any function. Then :

(i) If f is continuous at c, and also,

(ii) f is differentiable on $(c - r, c) \bigcup (c, c + r)$ for some $r > 0$, and

(iii) there is $\delta > 0$ with $\delta \leq r$ such that $f'(x) \leq 0 \quad \forall x \in (c - \delta, c)$ and $f'(x) \geq 0 \quad \forall x \in (c, c + \delta)$

then prove that f has a local minimum at c.

(10) Let $D \subseteq \mathbb{R}, c$ be an interior point of D, and $f : D \longrightarrow \mathbb{R}$ be any function. Then :

(i) If f is continuous at c, and also,

(ii) f is differentiable on $(c - r, c) \bigcup (c, c + r)$ for some $r > 0$, and

(iii) there is $\delta > 0$ with $\delta \leq r$ such that $f'(x) \geq 0 \quad \forall x \in (c - \delta, c)$ and $f'(x) \leq 0 \quad \forall x \in (c, c + \delta)$

then prove that f has a local maximum at c.

(11) Let $D \subseteq \mathbb{R}, c$ be an interior point of D, and $f : D \longrightarrow \mathbb{R}$ be any function. If f is twice differentiable at c with $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(12) Let $D \subseteq \mathbb{R}, c$ be an interior point of D, and $f : D \longrightarrow \mathbb{R}$ be any function. If f is twice differentiable at c with $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

(13) Let $D \subseteq \mathbb{R}, \; c$ be an interior point of D, and $f : D \to \mathbb{R}$ be a function. Let f be twice differentiable at c. If c is a point of inflection for f, then prove that $f''(c) = 0$.

(14) Let $D \subseteq \mathbb{R}, \; c$ be an interior point of D, and $f : D \to \mathbb{R}$ be a function such that f is thrice differentiable at c. If $f''(c) = 0$ and $f^3(c) \neq 0$, then prove that c is a point of inflection for f.

(15) State L'Hôpital's Rule for $\frac{0}{0}$ Indeterminate Form when $x \longrightarrow c, c \in \mathbb{R}$.

(16) State L'Hôpital's Rule for $\frac{0}{0}$ Indeterminate Forms when $x \longrightarrow \infty$.

(17) State L'Hôpital's Rule for $\frac{0}{0}$ Indeterminate Forms when $x \longrightarrow -\infty$.

3.7

(18) State L'Hôpital's Rule for $\frac{\infty}{\infty}$ Indeterminate Forms when $x \longrightarrow c, c \in \mathbb{R}$.

(19) State L'Hôpital's Rule for $\frac{\infty}{\infty}$ Indeterminate Forms when $x \longrightarrow \infty$.

(20) State L'Hôpital's Rule for $\frac{\infty}{\infty}$ Indeterminate Forms when $x \longrightarrow \infty$.

xxxxxxxxxxxxx

Chapter 4

(USMT 202) Discrete Mathematics

4.1 Practical 2.1: Finite, Infinite, Countable and Uncountable Sets, Counting Principles, Two Way Counting

4.1.1 Prerequisite of Practical 2.1

(1) A set S is said to be **finite** if it is empty or if there is a bijective function between S and $\mathbb{N}_m$ for some $m \in \mathbb{N}$ ($\mathbb{N}_m = \{1, 2, \cdots, m\}$).

(2) If there is a bijective function between a non-empty set S and $\mathbb{N}_m$ for some $m \in \mathbb{N}$ then we say that S has the size m or has the **cardinality** m and we write $|S| = m$. The cardinality of an empty-set is 0 that is $|\emptyset| = 0$ and we say that empty set has 0 elements.

(3) A set which is not finite is said to be **infinite**.

(4) **Note:**

   (i) The cardinality of $\mathbb{N}_n = n$ i.e. $|\mathbb{N}_n| = n$.

   (ii) $|X| = n$ if and only if there is a bijection between X and $\mathbb{N}_n$.

(5) If $S \neq \emptyset, S \subseteq \mathbb{N}_n$ for some $n \in \mathbb{N}$ then $|S| = m$ for some $m \in \mathbb{N}, m \leq n$.

(6) Let A be a nonempty set; let $n \in \mathbb{N}$. Then the following are equivalent:
(1) There is a surjective function $f : \mathbb{N}_n \longrightarrow A$.
(2) There is an injective function $g : A \longrightarrow \mathbb{N}_n$.
(3) A is finite and $|A| \leq n$.

(7) If $|X| = n$ and $S \subseteq X$ then $|S| \leq |X|$.

(8) If X, Y are finite sets and there is an injective function $f : X \longrightarrow Y$ then $|X| \leq |Y|$.

(9) The set $\mathbb{N}$ is infinite.

(10) If the set S is such that there is a bijection $b : \mathbb{N} \longrightarrow S$, then S is infinite. (converse not true.)

(11) If X is a subset of Y, and X is infinite, then Y is infinite.

(12) A set X is said to be **countable** if it is finite or if it is in bijection with $\mathbb{N}$.

(13) A set that is not countable is said to be **uncountable**.

(14) A set X that is both infinite and countable is said to be **countably infinite** or **denumerable**.

(15) The sets $\mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N}, \mathbb{Q}$ are countable.

(16) The sets $(0, 1), \mathbb{R} \setminus \mathbb{Q}, \mathbb{R}$ are uncountable.

(17) A subset of a countable set is also countable (that is finite or countably infinite)

(18) Let A be a nonempty set. Then the following are equivalent:
(1) There is a surjective function $f : \mathbb{N} \longrightarrow A$.
(2) There is an injective function $g : A \longrightarrow \mathbb{N}$.
(3) A is countable. (that is either finite or countably infinite).

(19) **Addition Principle** If A and B are non-empty finite sets, and X and Y are disjoint then $|A \cup B| = |A| + |B|$.

Note:

(i) The rule is still valid if X, or B, or both X and Y are empty.
(ii) The rule can be extended to the union of any number of pairwise disjoint sets $A_1, A_2, \cdots, A_n$.
$$|A_1 \cup A_2 \cup \cdots \cup A_n| = |A_1| + |A_2| + \cdots + |A_n|$$
(iii) If there are r_1 distinct objects in the first set, r_2 distinct objects in the second set, $\cdots$, and r_m distinct objects in the m^{th} set, and if the different sets are disjoint, then the number of ways to select an object from one of the m sets is $r_1 + r_2 + \cdots + r_m$.

(20) **Product Set**: Let X, Y be any sets. The product set $X \times Y$ is the set of all ordered pairs (x, y).

(21) **The Multiplication Principle**: Let X and Y be finite non-empty sets. Then the size of $X \times Y$ is given by $|X \times Y| = |X| \times |Y|$.

(22) **Two way counting theorem**: Let X and Y be finite non-empty sets, and S be a subset of $X \times Y$. Let $R_x(S)$ be the set of pairs in S whose first coordinate is x and $C_y(S)$ be the set of pairs in S whose second coordinate is y. Then the following results hold.

(i) The size of S is given by
$$|S| = \sum_x |R_x(S)| = \sum_y |C_y(S)|.$$
(ii) If $|R_x(S)|$ is a constant r, independent of x and $|C_y(S)|$ is a constant c, independent of y, then $r|X| = c|Y|$.

(A) Objective Questions

Choose correct alternative in each of the following:

(1) If X is a finite set and $f : \mathbb{N}_n \longrightarrow X$ is a surjective function for some $n \in \mathbb{N}$ then

(a) $|X| \leq n$. (b) $|X| > n$. (c) $|X| = n$ (d) None of these.

(2) If A is a countable set, and B is an uncountable set, then the most we can say about $A \cup B$ is that it is

(a) Finite (b) Countable (c) Uncountable (d) None of these

(3) If A is a countable set, and B is finite set, then the most we can say about $A \cup B$ is that it is

(a) Finite (b) Countable (c) Uncountable (d) None of these

(4) If A is an uncountable set, and B is finite set, then the most we can say about $A \cup B$ is that it is

(a) Finite (b) Countable (c) Uncountable (d) None of these

(5) If X is a countable set, and Y is an uncountable set, then the most we can say about the Cartesian product $X \times Y$ is that it is

(a) Finite (b) Countable (c) Uncountable (d) None of these

(6) If a set S is such that $\exists$ a bijection between S and $\mathbb{N}_s$ and there exists a bijection between S and $\mathbb{N}_t$ for some $s, t \in N$, then

(a) $s = t$ (b) $s \neq t$ (c) $s > t$ (d) $s < t$

(7) If X, Y are finite sets and there is an injective function $f : X \to Y$ then

(a) $|X| = |Y|$ (b) $|X| \leq |Y|$ (c) $|X| \geq |Y|$ (d) $|X| < |Y|$

(8) The number of functions from a set with m elements to one with n elements are

(a) m^n (b) n^m (c) $m \times n$ (d) None of these

(9) A new company with just two employees, Sanchez and Patel, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

(a) 23 (b) 132 (c) 144 (d) None of these

(10) In how many ways can we draw a heart or a spade from an ordinary deck of playing cards?

(a) 169 (b) 26 (c) 52 (d) None of these

(11) A student can choose a computer project from one of three lists. The three lists contain 23, 15 and 19 possible projects, respectively. How many possible projects are there to choose from ?

(a) 38 (b) 57 (c) 34 (d) 42

(12) There are 32 micro computers in a computer center. Each micro computer has 24 ports. How many different ports to a micro computer in the center are there ?

(a) 56 (b) 768 (c) 8 (d) None of these

(13) The number of ways to pick a sequence of two different letters of the alphabet that appear in the word BOAT is

(a) 21 (b) 12 (c) 8 (d) None of these

(14) The number of ways to pick first a vowel and then a consonant from the word MATHEMATICS is

(a) 56 (b) 15 (c) 4 (d) None of these

(15) How many ways are there to pick a man and a woman who are not husband and wife from a group of n married couples

(a) $n!$ (b) $n(n-1)$ (c) $n+(n-1)$ (d) None of these

(B) Descriptive Questions

(1) Show that in any set X of people there are two members of X who have the same number of friends in X.

(2) Show that the set $\mathbb{N}$ is infinite.

(3) Prove that if X is a subset of Y, and X is infinite, then Y is infinite.

(4) Show that the set $\mathbb{Z}$ of all integers is countable.

(5) Show that every infinite subset of $\mathbb{N}$ is countable.

(6) Prove that a subset of a countable set is countable.

(7) Show that $\mathbb{N} \times \mathbb{N}$ is countable.

(8) Show that $\mathbb{Q}^+$ is countable.

(9) Show that $\mathbb{Q}$ is countable.

(10) Show that $(0, 1)$ the set of all real numbers between 0 and 1 is uncountable.

(11) Show that $[0, 1]$ is uncountable.

(12) Show that $\mathbb{R}$ is uncountable.

(13) In Algebra class, 40 of the students are boys. Each boy knows six of the girls in the class and each girl knows eight of the boys. How many girls are in the class?

(14) How many different 4-letter radio station call letters (upper case) can be made
a) if the first letter must be a K or W and no letter may be repeated?
b) if repeats are allowed (but the first letter is a K or W).
c) How many of the 4-letter call letters (starting with K or W) with no repeats and ending in R?

(15) How many ways are there to pick 2 different cards from a standard 52 card deck such that:
(a) The first card is an Ace and the second card is not a Queen?
(b) The first card is a spade and the second card is not a Queen?

(16) How many ways are there to roll two dice to yield a sum divisible by 3?

(17) How many nonempty different collections can be formed from five (identical) apples and eight (identical) oranges?

(18) How many two-digit numbers have distinct and non-zero digits?

(19) How many ways can we get a sum of 4 or of 8 when two distinguishable dice (say red and white) are rolled? How many ways can we get an even sum?

(20) In how many ways can we draw a heart or a spade from an ordinary deck of playing cards ? A heart or an ace ? An ace or a king ? A card numbered 2 through 10 ? A numbered card or a king ?

(21) A store carries 8 styles of pants. For each style, there are 10 different possible waist sizes, 6 pants lengths, and 4 colour choices. How many different types of pants could the store have?

(22) Given eight different English books, seven different French books, and five different German books:
(a) How many ways are there to select one book?
(b) How many ways are there to select three books, one of each language?

(23) How many ways are there to form a three-letter sequence using the letters a, b, c, d, e, f:
(a) with repetition of letters allowed?
(b) without repetition of any letter?
(c) without repetition and containing the letter e?
(d) with repetition and containing e?

4.2

4.2 Practical 2.2: Stirling numbers of second kind, Pigeon hole principle

4.2.1 Prerequisite of Practical 2.2

(1) A **partition** of a set X is a family $\{X_i/i \in I\}$ of non-empty subsets of X such that

(i) X is the union of the sets $X_i \quad (i \in I)$.

(ii) each pair $X_i, X_j \quad (i \neq j)$ is disjoint.

The subsets X_i are called the **parts** of the partition.

(2) Let n, k be non-negative integers. The **Stirling number of second kind** $\mathbf{S}(n, k)$ is the total number of partitions on an $n-$ set into k disjoint, non-empty, unordered subsets.
Note:

(i) Equivalently, a Stirling number of the second kind can identify how many ways a number of distinct objects can be distributed among identical non-empty bins. Let n be the number of distinct objects to be distributed among k identical bins.

(ii) By convention, $S(0, 0) = 1$ while $S(n, 0) = 0$ for every positive integer n.

For example: Find $S(4, 2)$ by writing all the partitions of the set $\{1, 2, 3, 4\}$ into two parts.
All the partitions of $\{1, 2, 3, 4\}$ into two parts are

(1)	$\{1\}, \{2, 3, 4\}$;	(5)	$\{1, 2\}, \{3, 4\}$;
(2)	$\{2\}, \{1, 3, 4\}$;	(6)	$\{1, 3\}, \{2, 4\}$;
(3)	$\{3\}, \{1, 2, 4\}$;		
(4)	$\{4\}, \{1, 2, 3\}$;	(7)	$\{1, 4\}, \{2, 3\}$.

Hence $S(4, 2) = 7$

(3) $S(n, n-1) = \binom{n}{2}$.

(4) The number of ways of putting n balls of distinct colours into k distinct boxes with each box containing at least one ball is $k!S(n, k)$.

(5) Let n and k be positive integers. Then show that the number of surjective functions from an $n-$set to a $k-$set is equal to $k!S(n, k)$.

(6) Let n, k be positive integers with $n \geq k$. Then
.
$$S(n, k) = S(n-1, k-1) + k\ S(n-1, k).$$

(7) For all $n \geq 2$, $S(n, 2) = 2^{n-1} - 1$.

(8) **Simplest form of the Pigeonhole Principle**:
When $n + 1$ pigeons are to be put in n boxes, there is at least one box that receives two (or more) pigeons.

(9) **Strong pigeonhole principle**:
Let $q_1, q_2, \cdots, q_n$ be positive integers. If $q_1 + q_2 + \cdots + q_n - n + 1$ objects are put into n boxes, then either the first box contains at least q_1 objects, or the second box contains at least q_2 objects,$\cdots$, or the n^{th} box contains at least q_n objects.

Note:

(1) It is possible to distribute $q_1 + q_2 + \cdots + q_n - n$ objects among n boxes by putting $q_1 - 1$ objects into the first box, $q_2 - 1$ objects into the second box, and so on.

(2) The simple form of the pigeonhole principle is obtained from the strong form by taking $q_1 = q_2 = \cdots = q_n = 2 \quad \therefore q_1 + q_2 + \cdots + q_n - n + 1 = 2n - n + 1 = n + 1$ and the statement becomes if $n + 1$ objects are to be distributed into n boxes then either the first box contains at least $q_1 = 2$ objects or the second box contains at least $q_2 = 2$ objects $\cdots$ or the n^{th} box contains at least $q_n = 2$ objects. i.e. If $n + 1$ objects are put into n boxes, then at least one box contains two or more of the objects.

(3) If $q_1 = q_2 = \cdots = q_n = r+1 \quad \therefore q_1+q_2+\cdots+q_n-n+1 = (r+1)\,n-n+1 = r\,n+1$ then the statement becomes if $n\,r+1$ objects are to be distributed into n boxes then either the first box contains at least $q_1 = r+1$ objects or the second box contains at least $q_2 = r+1$ objects $\cdots$ or the n^{th} box contains at least $q_n = r+1$ objects.
In short, If $r\,n+1$ objects are put into n boxes, then at least one box contains $r+1$ or more of the objects.
$\therefore$ **If m objects are distributed into n boxes and $m \geq nr+1$, then at least one box contains at least $r+1$ objects.**

4.2.2 PRACTICAL 2.2

(A) Objective Questions

Choose correct alternative in each of the following:

(1) Which of the following is a partition of $\{1, 2, \ldots, 8\}$?

(a) $\{\{1,3,5\},\{1,2,6\},\{4,7,8\}\}$
(b) $\{\{1,3,5\},\{2,6,7\},\{4,8\}\}$
(c) $\{\{1,3,5\},\{2,6\},\{2,6\},\{4,7,8\}\}$
(d) $\{\{1,5\},\{2,6\},\{4,8\}\}$

(2) Let $S(n,k)$ denote Stirling number of second kind on n-set into k-disjoint nonempty unordered subsets, then $S(0,0)$ is

(a) 1 (b) 0 (c) n (d) None of these

(3) Let $S(n,k)$ denote Stirling number of second kind on n-set into k-disjoint nonempty unordered subsets, then $S(n,n)$ is

(a) 0 (b) 1 (c) n (d) None of these

(4) Let $S(n,k)$ denote Stirling number of second kind on n-set into k-disjoint nonempty unordered subsets, then $S(n,k) = 0$ if

(a) $k > n$ (b) $k < n$ (c) $k = n$ (d) None of these

(5) Let $S(n, k)$ denote Stirling number of second kind on n-set into k-disjoint nonempty unordered subsets, then $S(n, 1)$ is

(a) $n!$ (b) $(n-1)!$ (c) 1 (d) None of these

(6) If n and k be positive integers with $n \geq k$, then $S(n, k)$ has recurrence formula

(a) $S(n, k) = S(n-1, k-1) + kS(n, k)$ (c) $S(n, k) = S(n-1, k-1) + kS(n, k-1)$
(b) $S(n, k) = S(n-1, k-1) + kS(n-1, k)$ (d) None of these

(7) A basket of fruit is being arranged out of apples, bananas, and oranges. What is the smallest number of pieces of fruit that should be put in the basket in order to guarantee that either there are at least 8 apples or at least 6 bananas or at least 9 oranges?

(a) 12 (b) 21 (c) 20 (d) None of these

(8) What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

(a) 25 (b) 26 (c) 5 (d) None of these

(9) How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

(a) 101 (b) 102 (c) 100 (d) None of these

(10) The number of pigeons are distributed among k pigeonholes, then at least one pigeonhole contains two or more pigeons is

(a) $k+1$ or more (b) k or more (c) $k-1$ or more (d) None of these

(11) Let m objects be distributed into n boxes, then at least one box contains at least $r+1$ objects only if,

(a) $m > nr$ (b) $m < nr$ (c) $m = nr$ (d) None of these.

(B) Descriptive Questions

(1) Define Stirling number $S(n, k)$ of second kind. Prove that $S(n, n-1) = \binom{n}{2}$.

(2) Prove that $S(n, n-2) = \binom{n}{3} + 3\binom{n}{4} = \frac{1}{4}(3n-5)\binom{n}{3}$.

(3) Prove that $S(n, n-3) = \frac{1}{2}(n^2 - 5n + 6)\left(\begin{array}{c} n \\ 4 \end{array}\right)$.

(4) Without finding the value of either the LHS or RHS, show that $S(5,2) = \binom{5}{1} + \binom{5}{2}$.

(5) Find $S(7,3)$ by using the recursion formula for $S(n,k)$.

(6) Let $S(n,k)$ be the Stirling number of second kind. Find $S(4,2), S(5,2), S(6,2), S(6,3)$.

(7) Show that if we take $n+1$ numbers from the set $\{1, 2, \ldots, 2n\}$, then some pair of numbers will have no factors in common.

(8) Show that if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 3n\}$, then there are always two which differ by at most two.

(9) Given 5 points in the plane with integer coordinates, show that there exists a pair of points whose midpoint also has integer coordinates.

(10) During a month with 30 days a baseball team plays at least a game a day, but no more than 45 games. Show that there must be a period of some number of consecutive days during which the team must play exactly 14 games.

(11) A student has 6 weeks (that is, 42 days) to prepare for her examination and she has decided that during this period she will put in a total of 70 hours towards her preparation for the examination. She decides to study in full hours every day, studying at least one hour on each day. Prove that no matter how she schedules her studying pattern, she will study for exactly 13 hours during some consecutive days.

(12) A chess player has 77 days to prepare for a serious tournament. He decides to practice by playing at least one game per day and a total of 132 games. Show that there is a succession of days during which he must have played exactly 21 games.

(13) A chess master who has 11 weeks to prepare for a tournament decides to pay at least one game every day but, in order not to tire himself, he decides not to play more than 12 games during any calender week. Show that there exists a succession of (consecutive) days during which the chess master will have played exactly 21 games.

(14) Prove that if seven distinct numbers are selected from $\{1, 2, \ldots, 11\}$, then some two of these numbers sum to 12.

(15) From the integers $1, 2, \ldots, 200$, we choose 101 integers. Show that, among the integers chosen, there are two such that one of them is divisible by the other.

(16) Show that in any set of six people there are either three mutual friends or three mutual strangers. Further, show that it is possible to have a group of five people such that in any collection of three people out of these five, only two are mutual friends or only two are mutual strangers.

(17) Ten line segments are drawn, joining (1, 1), (7, 5), (8, 2), (9, 4) and (4, 4). Identify a mid point of these 10 line segments, such that both the coordinates of the mid-point are integers.
(Generalization of the above problem) Let $P_1, P_2, P_3, P_4 and P_5$ be any 5 lattice points in

the plane. (A point in the Cartesian plane is called a lattice point if both of its coordinates are integers). Show that at least one of the line segments, determined by these lattice points has some lattice point (not necessarily from P_1 to P_5) as its mid-point.

(18) Show that in any set of 10 people there are either four mutual friends or three mutual strangers.

(19) Is it possible to draw a regular pentagon along with all its diagonals, using two colours Red and Blue, such that there does not exist any triangle (created by considering any three points of the five points) which has all its sides either of Red colour or of Blue colour? (Draw such a pentagon and confirm that the answer to this question is affirmative. While joining the points, you may consider a continuous segment as Red colour and a dotted segment as Blue colour.)

(20) If 5 points are chosen at random in the interior of an equilateral triangle of side length 2 units, show that at least 1 pair of points has a separation of less than 1 unit.

(21) If 10 points are chosen at random in the interior of an equilateral triangle of side length 3 units, show that at least 1 pair of points has a separation of less than 1 unit.

(22) If 5 points are chosen at random in the interior of a square of side length 2 units, show that at least 1 pair of points has a separation of less than $\sqrt{2}$ units.

(23) Show that among any five points inside an equilateral triangle of side length 1, there exist two points whose distance is at most $\frac{1}{2}$.

4.3 Practical 2.3: Multinomial theorem, identities, permutation and combination of multi-set

4.3.1 Prerequisite of Practical 2.3

(1) **Binomial Theorem**:

Let n be a non-negative integer. Then

$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{n} y^n.$

(2) The number of ways of putting n distinct objects in r distinct boxes $B_1, B_2, \cdots, B_r$ such that the i^{th} box B_i holds n_i objects is called a **multinomial coefficient** and is denoted by $\binom{n}{n_1,\ n_2,\ \cdots, n_r}$.

(3) Let S be an $n-$ set and suppose the n objects in S are to be put in r distinct boxes $B_1, B_2, \cdots, B_r$ such that the i^{th} box B_i contains n_i objects with $n_1 + n_2 + \cdots + n_r = n$. Then the number of ways of doing this is equal to $\binom{n}{n_1,\ n_2,\ \cdots, n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}$.

(4) **The multinomial theorem:** Let n be a non-negative integer. Then :

$(x_1 + x_2 + \cdots + x_r)^n = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1,\ n_2,\ \cdots, n_r} {x_1}^{n_1} {x_2}^{n_2} \cdots {x_r}^{n_r},$

where the summation extends over all nonnegative integers $n_1, n_2, \cdots + n_r = n$.
We can use this in the following examples:

(i) Find the coefficient of ${x_1}^2 x_3 {x_4}^3 x_5$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^7$.
Coeffiecient of ${x_1}^2 x_3 {x_4}^3 x_5$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^7$ is $\dfrac{7!}{2!1!3!} = \dfrac{4*5*6*7}{2} = 420$.

(ii) Find the coefficient of ${x_1}^3 x_2 {x_3}^2$ in the expansion of $(2x_1 - 3x_2 + 5x_3)^6$.
Put $2x_1 = X_1, -3x_2 = X_2, 5x_3 = X_3$.
We will find coefficient of $X_1^3 X_2 X_3^2$ in $(X_1 + X_2 + X_3)^6$.
The coefficient $= \dfrac{6!}{3!1!2!} = \dfrac{4*5*6}{2} = 60$.
So the term in the expansion of $(2x_1 - 3x_2 + 5x_3)^6$ will be $60 * (2x_1)^3 (-3x_2)(5x_3)^2 = 60 * 8 * (-3) * 25 x_1^3 x_2 x_3^2$.
So the coefficient of ${x_1}^3 x_2 {x_3}^2$ is $-60 * 600 = -3600$.

(5) **Pascal Identity**: Let n and k be positive integers. Then $\dbinom{n}{r} = \dbinom{n-1}{r} + \dbinom{n-1}{r-1}$.

(6) Some more identities:

(i) $\displaystyle\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}$.

(ii) $\displaystyle\sum_{i=r}^{n} \binom{i}{r} = \binom{n+1}{r+1}$.

(iii) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

(iv) $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

(7) An $r-$**permutation** of a set S is an ordered $r-$tuple of elements of S.

Note:

(i) The number of $r-$permutations of an $n-$set is denoted by $P(n, r) = n(n-1)(n-2)\cdots(n-(r-1)) = \dfrac{n!}{(n-r)!}$.

(ii) $P(n, n) = n!$ and $P(n, 1) = n$.

(iii) $P(n, r) = 0$ if $r > n$.

(iv) An $n-$permutation of $\{1, 2, \ldots, n\}$ is called a permutation. And if $(i_1, i_2, \cdots, i_n)$ is a permutation, we can write it as $i_1 i_2 \ldots i_n$ or

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ i_1 & i_2 & i_3 & \cdots & i_n \end{pmatrix}.$$

4.3

(8) **Permutations of Multisets**: If S is a multi-set, an $r-$permutation of S is an ordered arrangement of r of the objects of S.
If the total number of objects of S is n (counting repetitions), then an $n-$permutation of S will also be called a permutation of S.
e.g. $S = \{a, a, b, c, c, c\}$, then $acbc, cbcc, abca$ are some of the $4-$permutations.
And $abccca, acbacc$ are some of the permutations or $6-$permutations (they are called permutations since all the six letters are used)

(9) Let S be a multi-set consisting of k distinct objects, each with infinite multiplicity. Then the total number of $r-$permutations of S is k^r.

(10) Let S be a multi-set with k distinct objects with finite repetition numbers $n_1, n_2, \ldots, n_k$ respectively. So the size of S is $n = n_1 + n_2 + \cdots + n_k$. Then the number of permutations of S equals $\dfrac{n!}{n_1!n_2!\cdots n_k!}$.

(11) **Circular Permutations** If instead of arranging the objects in a line, we arrange them in a circle, the number of circular permutations is smaller.
Suppose six children are marching in a circle. In how many different ways can they form their circle?
Since the children are moving, what matters are their positions relative to each other and not to their environment.
Clearly, two circular permutations are considered as the same if one can get the other by a rotation, that is, by a circular shift.
There are six linear permutations for each circular permutation.
Thus, there is a $6-$to1 correspondence between the linear permutations of six children and the circular permutations of the six children.
Therefore, to find the number of circular permutations, we divide the number of linear permutations by 6.
Hence, the number of circular permutations of the six children equals $\dfrac{6!}{6} = 5!$

(12) The number of circular $r-$permutations of a set of n elements is given by

$$\frac{P(n,r)}{r} = \frac{n!}{r * (n-r)!}.$$

In particular, the number of circular permutations (permutation means $n-$permutation) of n elements is $(n-1)!$.

(13) **Combinations of Sets**: Let S be a set of n elements. A combination of a set S is an unordered selection of the elements of S.

(14) Let r be a non-negative integer. By an $r-$**combination** of a set S of n elements, is an unordered selection of r of the n objects.

(15) For $0 \le r \le n$, $P(n,r) = r! \times \binom{n}{r}$.

(16) **Combinations of Multi-sets**: If S is a multi-set, then an $r-$combination of S is an unordered selection of r of the objects of S. Note: (1) Thus an $r-$combination of S is itself a multi-set, a submulti-set of S.

(2) If S has n objects, then there is only one $n-$combination of S,namely, S itself.
(3) If S contains objects of k different types, then the number of $1-$combinations of $S = k$

(17) Let S be a multi-set with objects of k different types each with an infinite repetition number (multiplicity). Then the number of $r-$combinations of S equals
$\binom{r+k-1}{r} = \binom{r+k-1}{k-1}$.

(18) The number of ways of putting r identical objects into k distinct boxes with each box containing at least one object is $\binom{r-1}{k-1}$.

4.3.2 PRACTICAL 2.3

(A) Objective Questions

Choose correct alternative in each of the following:

(1) How many 10-letter patterns can be formed from the letters of the word BASKETBALL?

(a) $C(10,10)$
(b) $\frac{10!}{2!2!1!1!1!1!2!}$
(c) $\frac{10!}{2!+2!+1!+1!+1!+1!+2!}$
(d) None of these

(2) In how many ways can the letters of the word 'LEADER' be arranged?

(a) 72 (b) 144 (c) 360 (d) None of these

(3) In how many ways can 15 billiard balls be arranged in a row if 3 are red, 4 are white and 8 are black?

(a) 12 (b) 18 (c) 96 (d) None of these

(4) In how many ways can a party of 9 persons arrange themselves around a circular table?

(a) 9! (b) 8! (c) 9!+8! (d) None of these

(5) The number of ways of placing 8 similar balls in 5 numbered boxes is

(a) $C(12,8)$ (b) $C(13,8)$ (c) $C(12,5)$ (d) None of these

(6) The number of terms in the expansion of $(2x+3y-5z)^8$ is

(a) $C(10,8)$ (b) $C(11,8)$ (c) $C(10,3)$ (d) None of these

(7) How many ways are there to select a captain and a vice captain from 15 members of a cricket team?

(a) $P(15,2)$ (b) $P(14,2)$ (c) $P(15,14)$ (d) None of these

(8) Suppose that Sachin has to visit eight different cities. He must begin his trip in a specified city, but later he can visit the other seven cities in any order he wishes. How many possible ways are there for Sachin to visit these cities?

(a) 7! (b) 8! (c) 56 (d) None of these

(9) How many ways are there to select five players for an inter college basket ball match from a 10-member team?

(a) $C(10,5)$ (b) $P(10,5)$ (c) $C(10,2)$ (d) None of these

(10) How many 3 digit numbers can be formed using the digits 1, 3, 5, 7, 9, where we are allowed to repeat the digits?

(a) 125 (b) 25 (c) 5 (d) None of these

(11) Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department of a college. How many ways are there to select a committee to develop a discrete mathematics course at the college if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

(a) $C(9,3).C(11,4)$ (b) $P(9,3).P(11,4)$ (c) $P(9,3).C(11,4)$ (d) None of these

(12) How many solutions are there to the equation $x_1 + x_2 + x_3 + x_4 = 17$, where x_1, x_2, x_3, and x_4 are nonnegative integers?

(a) $C(20,3)$ (b) $C(17,3)$ (c) $C(21,3)$ (d) None of these

(13) A box contains 12 black and 8 green marbles. How many ways can 3 black and 2 green marbles be chosen?

(a) $C(12,3) + C(8,2)$ (c) $C(12,5) + C(8,5)$
(b) $C(12,2) + C(8,3)$ (d) None of these

(B) Descriptive Questions

(1) Find the coefficient of $x_1^2 x_3 x_4^3 x_5$ in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^7$.

(2) Find the coefficient of $x_1^3 x_2 x_3^2$ in the expansion of $(2x_1 - 3x_2 + 5x_3)^6$.

(3) How many 11-letter words can be made from the letters of the word ABRACADABRA?

(4) How many 11 letter words can be made form the letters of the word MISSISSIPPI?

(5) Evaluate the multinomial numbers $\begin{pmatrix} 11 \\ 4,3,2,1 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ 5,2 \end{pmatrix}$.

(6) Prove that

(i) $\sum_{k=0}^{r}\begin{pmatrix} m \\ k \end{pmatrix}\begin{pmatrix} n \\ r-k \end{pmatrix} = \begin{pmatrix} m+n \\ r \end{pmatrix}$. (iv) $\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} = 2^n$.

(ii) $\sum_{i=r}^{n}\begin{pmatrix} i \\ r \end{pmatrix} = \begin{pmatrix} n+1 \\ r+1 \end{pmatrix}$. (v) $2^n = \displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k}3^{n-k}$.

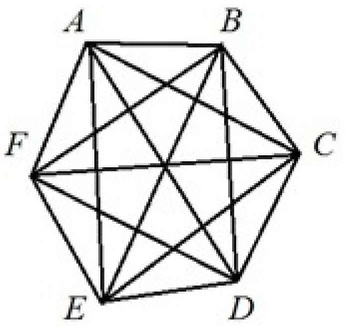
(iii) $\sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}^2 = \begin{pmatrix} 2n \\ n \end{pmatrix}$. (vi) $3^n = \displaystyle\sum_{k=0}^{n}\binom{n}{k}2^k$.

(7) Provide the combinatorial proof of the following identities:

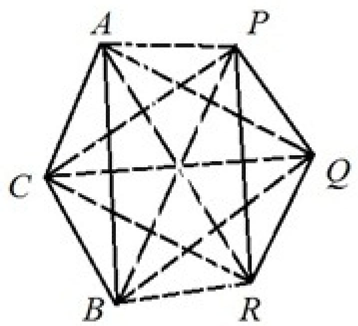
(i) $\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$.

(ii) $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

Hint:Observe the sketch, drawn below



ABCDEF is a hexagon



ABC and *PQR* are triangles

(8) How many r-permutations does an n-set have?

(9) If $S = \{a, b, c\}$, then find all 1-permutations, 2-permutations, 3-permutations of S.

(10) What is the number of ways to order the 26 letters of the alphabet so that no two of the vowels a, e, i, o, and u occur consecutively?

(11) How many seven-digit numbers are there such that the digits are distinct integers taken from $\{1, 2, \ldots 9\}$ and such that the digits 5 and 6 do not appear consecutively.

(12) Consider the multi-set $\{3.a, 2.b, 4.c\}$ of 9 objects of 3 types. Find the number of 8-permutations of S.

(13) Ten people, including two who do not wish to sit next to one another, are to be seated at a round table. How many circular seating arrangements are there?

(14) There are 15 people enrolled in a mathematics course, but exactly 12 attend on any given day. There are 25 seats in the classroom. Find the number of different ways in which an instructor might see the 12 students in the classroom.

(15) How many eight-letter words can be constructed by using the 26 letters of the alphabet if each word contains three, four or five vowels? It is understood that there is no restriction on the number of times a letter can be used in a word.

4.3

(16) Let S be the multi-set $\{10.a, 10.b, 10.c, 10.d\}$ with objects of four types ab, c and d. What is the number of 10-combinations of S which have the property that each of the four types of objects occurs at least once.

(17) Write the number of integral solutions of the equation $x_1 + x_2 + x_3 + x_4 = 20$ in which $x_1 \geq 3, x_2 \geq 1, x_3 \geq 0, x_4 \geq 5$

(18) There are five types of colour T-shirts on sale, black, blue, green, orange, and white. John is going to buy ten T-shirts, he has to buy at least two blues and two oranges, and at least one for all other colours. Find the number of ways that John can select ten T-shirts.

4.4 Practical 2.4: Inclusion-Exclusion principle. Euler phi function

4.4.1 Prerequisite of Practical 2.4

(1) **The Inclusion-Exclusion Principle**: Let X be a finite set and let $P_i;\ i = 1, 2, \ldots, n$ be a set of n properties satisfied by some of the elements of X. Let A_i denote the set of those elements of X that satisfy the property P_i. Then the size of the set $\overline{A_1} \cap \overline{A_2} \cdots \overline{A_n}$ of all those elements that do not satisfy any one of these properties is given by
$|\overline{A_1} \cap \overline{A_2} \cap \cdots \overline{A_n}| = |X| - \sum_{i=1}^{n} |A_i| + \sum_{1 \leq i < j \leq n} |A_i \cap A_j| - \cdots + (-1)^k \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3} \cdots \cap Ai_k| + \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n|$.

(2) A **derangement** of $\{1, 2, \ldots, n\}$ is a permutation $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ such that $i_1 \neq 1, i_2 \neq 2, \ldots, i_n \neq n$.
Thus a derangement of $\{1, 2, \ldots, n\}$ is a permutation $i_1 i_2 \ldots i_n$ of $\{1, 2, \ldots, n\}$ in which no integer is in its natural position. We denote the number of derangements of $\{1, 2, \ldots, n\}$ by D_n.
Note:

(i) For $n = 1$ there are no derangements. $D_1 = 0$

(ii) The only derangement for $n = 2$ is $(2, 1)$ i.e.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \therefore D_2 = 1$$

(iii) For $n = 4$, the different derangements are

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}.$$

$\therefore D_4 = 9$

(iv) We can show that $D_3 = 2$.

(3) The number of **derangements** D_n of $\{1, 2, \ldots, n\}$ is given by $D_n = n!\left\{\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \cdots + (-1)^n \frac{1}{n!}\right\}$.

(4) $e^{-1} = \frac{D_n}{n!}$.

(5) The number of integers x in the range $1 \leq x \leq n$ which are coprime to n, is denoted by $\phi(n)$, the value of **Euler's function** ϕ at n.

(6) Let $n \geq 2$ be an integer whose prime factorization is $n = p_1{}^{\alpha_1} p_2{}^{\alpha_2} \cdots p_r{}^{\alpha_r}$ where $\alpha_i \geq 1 \ \forall \ i, 1 \leq i \leq r$. Then $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_r}\right)$.

4.4.2 PRACTICAL 2.4

(A) Objective Questions

Choose correct alternative in each of the following:

(1) If a school has 100 students with 50 students taking French, 40 students taking Latin, and 20 students taking both languages, how many students take no language?

(a) 110 (b) 30 (c) 210 (d) None of these

(2) Suppose that there are 1807 freshmen at your school. Of these, 453 are taking a course in computer science, 567 are taking a course in mathematics, and 299 are taking courses in both computer science and mathematics. How many are not taking a course either in computer science or in mathematics?

(a) 1086 (b) 721 (c) 1020 (d) None of these

(3) How many positive integers not exceeding 1000 are divisible by 7 or 11?

(a) 232 (b) 220 (c) 244 (d) None of these

(4) Which is the following derangement of on $1, 2, 3, 4, 5$?

(a) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 4 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ (d) None of these

(5) At a party there are n men and n women. In how many ways can the n women choose male partners for the dance?

(a) D_n (b) $n!$ (c) $(n-2)!$ (d) None of these

(6) At a party there are n men and n women. How many ways are there for the dance if everyone has to change partners?

(a) $n!$ (b) D_n (c) $(n-2)!$ (d) None of these

(7) At a party, seven gentlemen check their hats. In how many ways can their hats be returned so that no gentleman receives his own hat?

(a) 7! (b) D_7 (c) $\frac{D_7}{7}$ (d) None of these

(8) At a party, seven gentlemen check their hats. In how many ways can their hats be returned so that at least one of the gentlemen receives his own hat?

(a) 7! (b) $7! - D_7$ (c) $7! \times D_7$ (d) None of these

(9) If p is a prime and $k > 0$, then:

(a) $\phi(p^k) = p^k\left(1 + \frac{1}{p}\right)$

(b) $\phi(p^k) = p\left(1 - \frac{1}{p}\right)$

(c) $\phi(p^k) = p^k\left(1 - \frac{1}{p}\right)$

(d) None of these

(10) If $n \geq 2$ is an integer whose prime factorisation is $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ where $\alpha_i \geq 1$, $\forall i, 1 \leq i \leq r$, then

(a) $\phi(n) = n\left(1 + \frac{1}{p_1}\right)\left(1 + \frac{1}{p_2 1}\right)\dots\left(1 + \frac{1}{p_r}\right)$

(b) $\phi(n) = \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2 1}\right)\dots\left(1 - \frac{1}{p_r}\right)$

(c) $\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2 1}\right)\dots\left(1 - \frac{1}{p_r}\right)$

(d) None of these

(11) For $n > 2$, $\phi(n)$ is:

(a) Prime number (b) Even number (c) Odd number (d) None of these

(12) If $n > 1$, is prime. Then $\phi(n)$ is:

(a) $n - 1$ (b) n (c) $n + 1$ (d) None of these

(13) $\phi(13)$ is:

(a) 13 (b) 12 (c) 14 (d) None of these

(B) Descriptive Questions

(1) There are 73 students in the first year Humanities class at the University of California. Among them a total of 52 can play the piano, 25 can play the violin, and 20 can play the flute, 17 can play both piano and violin, 12 can play piano and flute, and 7 can play violin and flute, but only one student can play all three instruments. How many in the class cannot play any of them?

(2) In a class of 67 mathematics students, 47 can read French, 35 can read German and 23 can read both languages. How many can read neither language? If, furthermore, 20 can read Russian, of whom 12 also read French, 11 read German also and 5 read all three languages, how many cannot read any of the three languages?

(3) Find the number of ways of arranging the letters A, E, M, O, U, Y in a sequence in such a way that the words ME and YOU do not occur.

(4) Define derangement.Write formula for derangement D_n and hence find D_5.

(5) Define Euler ϕ function. Find $\phi(60)$ by using Euler formula $\phi(n)$?

(6) A total of 1232 students have taken a course in Spanish, 879 have taken a course in French, and 114 have taken a course in Russian. Further, 103 have taken courses in both Spanish and French, 23 have taken courses in both Spanish and Russian, and 14 have taken courses in both French and Russian. If 2092 students have taken at least one of Spanish, French, and Russian, how many students have taken a course in all three languages?

(7) How many positive integers < 70 are relatively prime to 70?

(8) Suppose there are 100 students in a school and there are 40 students taking each language, French, Latin, and German. Twenty students are taking only French, 20 only Latin, and 15 only German. In addition, 10 students are taking French and Latin. How many students are taking all three languages? No language?

(9) Find the number of integers between 1 and 1000, inclusive, that are not divisible by 5, 6, and 8.

(10) How many permutations of the letters M, A, T, H, I, S, F, U, N are there such that none of the words MATH, IS, and FUN occur as consecutive letters? (Thus, for instance, the permutation MATHISFUN is not allowed, nor are the permutations INUMATHSF and ISMATHFUN.)

(11) Find the number of permutations $i_1, i_2, \ldots, i_n$ of $\{1, 2, \ldots, n\}$ in which 1 is not in the first position (i.e.$i_1 \neq 1$).

(12) Show that $e^{-1} = \frac{D_n}{n!}$ where D_n is the number of derangements on n symbols.

(13) Define derangement D_n. Show that $D_n = (n-1)(D_{n-2} + D_{n-1})$ $(n = 3, 4, 5, \ldots)$.

(14) Define derangement D_n. Show that $D_n = nD_{n-1} + (-1)^n$ $(n = 2, 3, 4, \ldots)$ with $D_1 = 0, D_2 = 1$.

(15) How many solutions does $x_1 + x_2 + x_3 = 11$ have, where x_1, x_2 and x_3 are non-negative integers $x \leq 3$, $x_2 \leq 4$ and $x_3 \leq 6$?

(16) If p is a prime and $k > 0$, then prove that $\phi(p^k) = p^k \left(1 - \frac{1}{p}\right)$.

(17) Define Euler ϕ function. Hence find $\phi(360)$.

(18) In how many ways 5 gents and 4 ladies dine at a round table, if no two ladies are to sit together?

4.5

(19) Twelve persons are made to sit around a round table. Find the number of ways they can sit such that 2 specified are not together.

4.5 USMT 202: Practical 2.5: Permutations, cycles and signature of Permutations

4.5.1 Prerequisite for Practical 2.5

(1) A **permutation** on $\mathbb{N}_n$ is a bijection from $\mathbb{N}_n$ onto itself.
If σ is a permutation on $\mathbb{N}_n$, then we write it as

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(n) \end{pmatrix}$$

where $1 \longrightarrow \sigma(1), 2 \longrightarrow \sigma(2), \dots$

(2) The **identity permutation** on $\mathbb{N}_n$ is denoted by

$$i = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix}.$$

(3) The set of all permutations on n symbols is denoted by S_n.

(4) Since the number of permutations on n symbols is $n!$, therefore $|S_n| = n!$.

(5) For every permutation $\sigma \in S_n$, we can find its inverse σ^{-1} using the following trick:
If $\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(n) \end{pmatrix}$ then $\sigma^{-1} = \begin{pmatrix} \sigma(1) & \sigma(2) & \sigma(3) & \dots & \sigma(n) \\ 1 & 2 & 3 & \dots & n \end{pmatrix}$.
Then we rearrange the columns so that the first row is again $1\ 2\ \dots\ n$.
For example: $\rho = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 3 & 2 & 8 & 7 & 6 & 1 \end{pmatrix} \in S_8$. Then
$\rho^{-1} = \begin{pmatrix} 4 & 5 & 3 & 2 & 8 & 7 & 6 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 3 & 1 & 2 & 7 & 6 & 5 \end{pmatrix}$ We will verify that $\rho \circ \rho^{-1} = i$ (i is the identity permutation in S_8).
$\rho \circ \rho^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 3 & 2 & 8 & 7 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 4 & 3 & 1 & 2 & 7 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$.

(6) **Cyclic notation**: Consider a permutation $\sigma_1 \in S_3$ as follows:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Under σ_1, 1 goes to 2 then 2 goes to 3 and then 3 goes back to 1. So, instead of writing this permutation in two rows we can write it as simply $(1\ \ 2\ \ 3)$.
Similarly the permutation $\sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \in S_3$ can be written as a cycle $(1\ \ 3\ \ 2)$.
Now we will write $\tau_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ as a cycle $(2\ \ 3)$. Since 1 goes to itself, we don't write it at all.

Length of a cycle: If $\sigma \in S_n$ is a cycle such that $\sigma = (a_1 \ \ a_2 \ \ \cdots \ \ a_r)$ then we say that σ is a cycle of length r.
So, τ_1 is a cycle of length 2.
$\tau_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1 \ \ 3) \quad \therefore \tau_2$ is a cycle of length 2.
$\tau_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1 \ \ 2) \quad \therefore \tau_3$ is a cycle of length 2.
A cycle of length 2 is called as a **transposition.**
Thus $\tau_1 = (2 \ \ 3)$, $\tau_2 = (1 \ \ 3)$ and $\tau_3 = (1 \ \ 2)$ are transpositions from S_3.
Remember that the transposition $(n \ \ m) = (m \ \ n)$.
Now, if it is given that $(3 \ \ 2 \ \ 5) \in S_5$ then the actual permutation is $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 2 & 4 & 3 \end{pmatrix}$.
Note that the numbers which are not present in the cycle, go to themselves.

Multiplication of two or more cycles:
Consider two cycles $\sigma = (1 \ \ 3 \ \ 5 \ \ 4)$ and $\tau = (2 \ \ 7 \ \ 6)$ from S_8.
We want to find $\sigma \circ \tau$.
One way of doing this is to express σ and τ in their standard forms and then find the required product i.e. $\sigma \circ \tau$.
$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 5 & 1 & 4 & 6 & 7 & 8 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 7 & 3 & 4 & 5 & 2 & 6 & 8 \end{pmatrix}$.

$$\begin{aligned} \sigma \circ \tau &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 5 & 1 & 4 & 6 & 7 & 8 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 7 & 3 & 4 & 5 & 2 & 6 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 5 & 1 & 4 & 2 & 6 & 8 \end{pmatrix}. \end{aligned}$$

We can also find the product without actually writing the given permutations in their standard form. $\sigma \circ \tau = (1 \ \ 3 \ \ 5 \ \ 4) \circ (2 \ \ 7 \ \ 6) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 5 & 1 & 4 & 2 & 6 & 8 \end{pmatrix}$.

(7) Every permutation in S_n can be expressed as a product of disjoint cycles.
For example: $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 5 & 1 & 4 & 2 & 6 & 8 \end{pmatrix} = (1 \ \ 3 \ \ 5 \ \ 4) \circ (2 \ \ 7 \ \ 6)$.
Note: A permutation can be a single cycle.

(8) Product of disjoint cycles is commutative.

(9) Every permutation is a product of disjoint cycles. This product is unique up to the order in which the cycles appear.

(10) We will see how to find Inverse of a cyclic permutation:
Consider $(1 \ \ 3 \ \ 5 \ \ 4) \in S_8$. We want to find $(1 \ \ 3 \ \ 5 \ \ 4)^{-1}$
Just reverse the order. The answer is $(4 \ \ 5 \ \ 3 \ \ 1)$.
Check that $(1 \ \ 3 \ \ 5 \ \ 4) \circ (4 \ \ 5 \ \ 3 \ \ 1) = i$ the identity permutation in S_8.
To find Inverse of a product of cycles:
Suppose we want to find inverse of $(1 \ \ 3 \ \ 8 \ \ 5 \ \ 4)(2 \ \ 3 \ \ 6 \ \ 4 \ \ 7 \ \ 5) \in S_8$
We want to find $[(1 \ \ 3 \ \ 8 \ \ 5 \ \ 4)(2 \ \ 3 \ \ 6 \ \ 4 \ \ 7 \ \ 5)]^{-1}$

Method 1 Write the product of the two cycles in the standard form.

$$(1\ 3\ 8\ 5\ 4)(2\ 3\ 6\ 4\ 7\ 5) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 2 & 1 & 4 & 5 \end{pmatrix}.$$

$$\begin{aligned} [(1\ 3\ 8\ 5\ 4)(2\ 3\ 6\ 4\ 7\ 5)]^{-1} &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 8 & 6 & 7 & 2 & 1 & 4 & 5 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 3 & 8 & 6 & 7 & 2 & 1 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 1 & 7 & 8 & 3 & 4 & 2 \end{pmatrix} \end{aligned}$$

Method 2 Use the property that $(\sigma \circ \tau)^{-1} = \tau^{-1} \circ \sigma^{-1}$ for all $\sigma, \tau \in S_n$.

$$\begin{aligned} [(1\ 3\ 8\ 5\ 4)(2\ 3\ 6\ 4\ 7\ 5)]^{-1} &= (2\ 3\ 6\ 4\ 7\ 5)^{-1}(1\ 3\ 8\ 5\ 4)^{-1} \\ &= (5\ 7\ 4\ 6\ 3\ 2)(4\ 5\ 8\ 3\ 1) \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 5 & 1 & 7 & 8 & 3 & 4 & 2 \end{pmatrix} \end{aligned}$$

(11) Inverse of a transposition is itself, that is, if $(p\ \ q) \in S_n$ then $(p\ \ q)^{-1} = (q\ \ p) = (p\ \ q)$

(12) A cycle $(a_1\ \ a_2\ \ a_3\ \ \cdots\ \ a_r)$ of length r, can be written as $(r-1)$ transpositions in S_n as follows:

$$\begin{aligned} (a_1\ \ a_2\ \ a_3\ \ \cdots\ \ a_r) &= (a_1\ \ a_r)(a_1\ \ a_{r-1})(a_1\ \ a_{r-2})\cdots(a_1\ \ a_2) \\ \text{e.g.}\quad (1\ 3\ 5\ 4) &= (1\ \ 4)(1\ \ 5)(1\ \ 3) \end{aligned}$$

This representation is not unique because, we can write

$$\begin{aligned} (a_1\ \ a_2\ \ a_3\ \ \cdots\ \ a_r) &= (a_1\ \ a_r)(a_1\ \ a_{r-1})(a_1\ \ a_{r-2})\cdots(a_1\ \ a_2) \\ &= (a_1\ \ a_r)(a_1\ \ a_{r-1})(a_1\ \ a_{r-2})\cdots(a_1\ \ a_2)(x\ \ y)(x\ \ y) \end{aligned}$$

for any $x, y \in \mathbb{N}_n$, since $(x\ \ y)(x\ \ y) = i$ the identity permutation.

(13) Thus every permutation can be expressed as a product of disjoint cycles and every cycle can be expressed as a product of transpositions and hence every permutation can be expressed as a product of transpositions.

(14) If a permutation is written as a product of r transpositions and it is also written as a product of r' transpositions then either r and r' are both even or both odd.

(15) If $\sigma \in S_n$ i.e. σ is a permutation on n symbols say $\{1, 2, \cdots, n\}$ then **signature** of σ is denoted by sgn of σ and is given by $\text{sgn}(\sigma) = \prod_{1 \le i < j \le n} \dfrac{\sigma(i) - \sigma(j)}{i - j}$ where $\prod$, denotes the product.

(16) If σ is any permutation in S_n, then the sign of σ is ± 1.

(17) A permutation σ of S_n is said to be an **odd permutation** if sgn $\sigma = -1$ and σ is said to be an **even permutation** if sgn $\sigma = 1$.

(18) A transposition is an odd permutation. i.e. sgn $(a \;\; b) = -1$.

(19) A cycle $(a_1 \;\; a_2 \;\; a_3 \;\; \cdots \;\; a_r)$ of length r can be written as a product of transpositions as follows.
$(a_1 \;\; a_2 \;\; a_3 \;\; \cdots \;\; a_r) = (a_1 \;\; a_r)(a_1 \;\; a_{r-1})(a_1 \;\; a_{r-2}) \cdots (a_1 - a_2)$.
Thus a cycle of length r is written as a product of $r-1$ transpositions.
Since sgn $(\sigma \circ \tau) =$ sgn $\sigma *$ sgn τ, we have,

$$\begin{aligned} \text{sgn}\Big((a_1 \;\; a_2 \;\; a_3 \;\; \cdots \;\; a_r)\Big) &= \text{sgn}\Big((a_1 \;\; a_r)(a_1 \;\; a_{r-1})(a_1 \;\; a_{r-2}) \cdots (a_1 - a_2)\Big) \\ &= \text{sgn}\Big((a_1 \;\; a_r)\Big) * \text{sgn}\Big((a_1 \;\; a_{r-1})\Big) * \text{sgn}\Big((a_1 \;\; a_{r-2})\Big) * \cdots * \text{sgn}\Big((a_1 \;\; a_2 \\ &= (-1) * (-1) * \cdots * (-1) \quad (r-1 \;\; \text{times})(\text{sgn of a transposition} = -1) \\ &= (-1)^{r-1} \end{aligned}$$

So, signature of a cycle of length r is $(-1)^{r-1}$.

(20) Cycles of even length are odd permutations and cycles of odd length are even permutations.

(21) Signature of the identity permutation is 1.

(22) sgn $\sigma =$ sgn $\sigma^{-1} \qquad \forall\, \sigma \in S_n$.

(23) If a permutation $\sigma \in S_n$ can be written as a product of r transpositions and also a product of r' transpositions , then either r and r' are both even or r and r' are both odd. (i.e. r and r' have the same parity)

(24) For any integer $n \geq 2$, exactly half of the permutations in S_n are odd and half are even.

(25) The set of all even permutations from S_n is denoted by A_n. Clearly $|A_n| = \dfrac{n!}{2}$.

4.5.2 PRACTICAL 2.5

(A) Objective Questions

Choose correct alternative in each of the following:

(1) Let $\mathbb{N}_n = \{1, 2, \cdots, n\}$ for a positive integer n then, $f : \mathbb{N}_n \to \mathbb{N}_n$ is a permutation if

(a) f is one one but not onto.
(c) f is onto but not one one.
(b) f is one one and onto.
(d) f is any function.

(2) What is the number of even permutations in S_3?

(a) 3 (b) 4 (c) 0 (d) 6

(3) The number of elements in S_n is

(a) n (b) $n!$ (c) $\frac{(n-1)!}{2}$ (d) 2^n

(4) If $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 1 & 2 & 4 & 5 & 6 & 7 \end{pmatrix} \in S_7$ then α^{-1} is

(a) (1 2 3). (b) (1 2)(3 6 7). (c) (1 2 3)(5 6 7). (d) (4 5).

(5) If $\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 5 & 6 & 3 & 2 \end{pmatrix} \in S_6$ then, number of disjoint cycles in the expression of μ as their product is

(a) 1 (b) 2 (c) 3 (d) 4

(6) How many permutations of S_4 are expressed as composite of disjoint 2-cycles?

(a) 12 (b) 6 (c) 3 (d) 10

(7) Signature of identity permutation of S_n is

(a) -1 (b) 0 (c) 1 (d) depends on n.

(8) For any integer $n \geq 2$, in S_n, the number of even permutations is

(a) $\frac{n}{2}$ (b) $\frac{n!}{2}$ (c) $\frac{n!}{4}$ (d) none of the above.

(9) If $\sigma = (1\ 2\ 3)(2\ 3)$ then σ^{-1} is

(a) (3 2 1)(3 2). (b) (1 2). (c) (2 3). (d) (1 3).

(10) If $\sigma = (2\ 5\ 3\ 4)$ is in S_6 then, $\sigma^k = I_6$ for what value of k

(a) 5 . (b) 2. (c) 3. (d) 4.

(11) If $\sigma \circ \tau^{-1}$ is an odd permutation then,

(a) Both σ, τ are odd. (b) Both σ, τ are even. (c) only if σ is odd and τ is even (d) one of the σ, τ is odd and other is even.

(12) If σ is odd, then which of the following is true?

(a) σ^2 is even and σ^3 is odd. (b) σ^2 is odd and σ^3 is even. (c) both σ^2 and σ^3 are odd. (d) both σ^2 and σ^3 are odd.

(13) How many transpositions does S_7 has?

(a) $\frac{7!}{2}$ (b) 14 (c) 21 (d) 42

(14) How many $\sigma \in S_4$ satisfy the equation $\sigma^2 = \sigma$?

(a) 1 (b) 6 (c) 9 (d) 12

(15) Signature of a cycle of length r is

(a) $(-1)^{(r)}$ (b) $(-1)^{(r-1)}$ (c) $(-1)^{(r+1)}$ (d) r

(16) If a permutation $\sigma \in S_n$ Sn can be written as a product of r transpositions and also a product of r' transpositions, then

(a) $r - r'$ even (b) $r - r'$ odd (c) $r - r' = 0$ (d) $|r - r'| = 1$

(17) Which of the following statements are true for A_n?

(i) signature of every element of A_n is 1.
(ii) product of any two elements of A_n is an element of A_n.
(iii) Identity element belongs to A_n.
(iv) inverse of every element of A_n belongs to A_n.

(a) All the four statements are true for even integer n.
(b) All the four statements are true for all integers n.
(c) i, ii are true but iii, iv are not true.
(d) i, ii and iii are true.

(18) If A_n be the set of all even permutations in S_n, then cardinality of A_n is

(a) always an even positive integer.
(b) an even positive integer for $n > 3$.
(c) is even only if n is even.
(d) is always an odd positive integer.

(19) The number of elements in A_6 is

(a) 6 (b) 720 (c) 360 (d) 2^6

(B) Descriptive Questions

(1) Write down all permutations on 3 symbols $\{1, 2, 3\}$.

(2) For the following permutations $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 5 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 3 & 5 \end{pmatrix}$, $\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}$,

(i) Show that $\alpha\beta \neq \beta\alpha$

(ii) Verify $\alpha(\beta\gamma) = (\alpha\beta)\gamma$

(iii) Verify $(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$

(3) Write each of the following permutations $\sigma = (24)(179)(48)$ and $\tau = (164)(1379)(83)$ from S_9 as a product of disjoint cycles and also find the product $\sigma o \tau$.

(4) Find the inverse of $\sigma = (1358)(2314)(67) \in S_8$ and verify that $\sigma o \sigma^{-1} = i$ where i is the identity permutation in S_8.

(5) Find the inverse of each of the following permutations. Verify it is the inverse by computing the product and showing it is the identity permutation.

(i) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$ (ii) $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 5 & 7 & 3 & 8 & 2 & 6 \end{pmatrix}$

(6) Define an even permutation. Express $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix} \in S_8$ as a product of disjoint cycles. Determine whether σ is odd or even.

(7) Let $\alpha = (1325)(143)(25) \in S_5$ Find α^{-1} and express it as a product of disjoint cycles. State whether $\alpha^{-1} \in A_5$.

(8) Find the signature of following permutation using the definition of signature. $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$

(9) Express the following as product of disjoint cycles.

(i) $\sigma = \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ in S_3.

(ii) $\sigma = \begin{pmatrix} 4 & 3 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 5 & 3 & 4 \end{pmatrix}$ in S_5.

(iii) $\sigma = \begin{pmatrix} 1 & 4 \end{pmatrix} \begin{pmatrix} 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}$ in S_4.

(10) Find $\sigma \circ \tau, \tau \circ \sigma, \tau^2, \sigma^{-1} \circ \tau^2$ for $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 1 & 6 & 5 & 3 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 1 & 6 & 2 & 4 \end{pmatrix}$.

(11) Verify that $(\sigma \circ \tau)^{-1} = \tau^{-1} \circ \sigma^{-1}$ for $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 1 & 6 & 4 & 3 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 6 & 1 & 4 \end{pmatrix}$.

(12) Find $x \in S_5$ such that $\sigma \circ x = \tau$ where $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 3 & 4 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$.

(13) Write all permutations of S_3 and list all even permutations of S_3.

(14) State whether the given permutations from S_5 are odd or even?

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 2 & 3 & 1 \end{pmatrix}$.

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix}$.

(iii) $(2\ 3)(1\ 4\ 3)(5\ 3)(1\ 3\ 2\ 5)$.

(iv) $(2\ 5)(1\ 3\ 2\ 4)(5\ 2\ 1)(3\ 4)$.

(15) Find the number of transpositions, 3-cycles, 4-cycles in S_4. Find the remaining permutations that are not cycles but composite of cycles in S_4. Further classify them into even and odd permutations.

(16) If $\alpha = (1\ 3\ 6\ 2\ 4)(5\ 8\ 7)(9), \beta = (1\ 5\ 8\ 6\ 2)(3\ 9\ 4)(7)$ and $\sigma = (1)(2\ 6\ 8\ 9\ 7\ 4)(3\ 5)$ then, show that $\sigma\alpha\sigma^{-1} = \beta$.

(17) Show that $(1\ 6)(1\ 3)(2\ 7)(2\ 5)(2\ 4) = (1\ 5)(3\ 5)(3\ 6)(5\ 7)(1\ 4)(2\ 7)(1\ 2)$. Find third representation of this permutation as product of transpositions that is distinct from the given two representations.

(18) State whether following permutations in S_6 are odd or even.

(i) (235)(132)(13)(24513)(612435) (ii) (123)(345)(12)(1345)(3512)

(19) Write down all permutations on 4 symbols $\{1, 2, 3, 4\}$. Hence write elements of A_4.

(20) Write the following permutations as product of disjoint cycles. Express as product of transpositions and find their signature.

(i) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 7 & 1 & 2 & 8 & 3 & 9 & 6 \end{pmatrix}$ (iii) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 4 & 6 & 7 & 8 & 3 & 2 & 9 & 5 \end{pmatrix}$

(ii) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 3 & 6 & 5 & 4 & 2 & 9 & 8 & 1 \end{pmatrix}$

(21) Write the following permutation, σ as product of disjoint cycles and find σ^{-1}, σ^2 for the given σ. Write whether $\sigma, \sigma^{-1}, \sigma^2$ are even or odd.

(i) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 5 & 1 & 2 & 3 & 8 & 9 & 6 & 7 \end{pmatrix}$ (ii) $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 3 & 7 & 5 & 4 & 8 & 2 & 1 \end{pmatrix}$

(iii) $\sigma = \begin{pmatrix} 1 & 3 & 5 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 4 & 1 \end{pmatrix}$ in S_5.

(22) Show that $(3\ 4\ 7\ 2)$ can be expressed as product of some transpositions of the form $(i, i+1) \in S_7$.

4.6 Practical 2.6: Recurrence Relations

4.6.1 Prerequisite of Practical 2.6

(1) Let $h_0, h_1, \ldots, h_n \ldots$ be a sequence of numbers.This sequence is said to satisfy a **linear recurrence relation** of order k, provided that there exist quantities $a_1, a_2, \ldots, a_k$, with $a_k \neq 0$, and a quantity b_n such that

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} + b_n, \quad (n \geq k).$$

For example:

(i) The sequence of **derangement** numbers $D_0, D_1, \cdots, D_n, \cdots$ satisfies the two recurrence relations

$$D_n = (n-1)D_{n-1} + (n-1)D_{n-2}, \quad (n \geq 2)$$
$$D_n = nD_{n-1} + (-1)^n, \quad (n \geq 1)$$

(ii) The **Fibonacci sequence** $f_0, f_1, f_2, \cdots, f_n, \cdots$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2} \quad (n \geq 2)$ of order 2 with $a_1 = 1, a_2 = 1$ and $b_n = 0$.

(iii) The **factorial sequence** $h_0, h_1, \ldots, h_n, \ldots,$ where $h_n = n!$ satisfies the recurrence relation $h_n = nh_{n-1} \quad (n \geq 1)$ of order 1 with $h_0 = 0, h_1 = h_2 = 1$.

(iv) The **geometric sequence** $h_0, h_1, \ldots, h_n, \ldots,$ where $h_n = r^n$ satisfies the recurrence relation $h_n = rh_{n-1} \quad (n \geq 1)$ of order 1 with $a_1 = r$ and $b_n = 0$.

Note: The quantities $a_1, a_2, \ldots, a_k$ may be constant or may depend on n. Similarly, the quantity b_n may be a constant or also may depend on n.

(2) Let $h_0, h_1, \ldots, h_n \ldots$ be a sequence of numbers.This sequence is said to satisfy a **linear homogeneous recurrence relation** of order k, provided that there exist quantities $a_1, a_2, \ldots, a_k$, with $a_k \neq 0$ such that

$$h_n = a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k}, \quad (n \geq k).$$

The linear homogeneous recurrence relation is said to have constant coefficients provided that $a_1, a_2, \ldots, a_k$ are constants.

(3) Solving Recurrence Relations:
There are different techniques to solve a recurrence relation. Two of them are as follows:
(1) Iteration method or Backtracking.
(2) Characteristic roots.

For example:

(i) Solve the Recurrence Relation:
$a_n = -2a_{n-1}; n \geq 2, a_1 = 3$ using iteration method.
Solution:

$$\begin{aligned}
a_n &= -2a_{n-1} \\
&= -2(-2a_{n-2}) \\
&= (-2)^2(-2a_{n-3}) \\
&= (-2)^3(a_{n-3}) \\
&= (-2)^k(a_{n-k}) \quad \text{for some k} \\
&\text{put } k = n-1 \\
&= 3(-2)^{n-1}
\end{aligned}$$

(ii) Solving a homogeneous recurrence relation of second degree using algebraic method: (Characteristic Roots)
Note: We are going to discuss a method for solving linear homogeneous recurrence relations with constant coefficients that is, recurrence relations of the form

$$h_n = a_1 h_{n-1} + a_2 h_{n-2}. \quad (n \geq 2) \qquad (*)$$

where a_1, a_2 are constants and $a_2 \neq 0$.

The polynomial equation $x^k - a_1 x^{k-1} - a_2 x^{q-2} - \cdots - a_k = 0,$ is called the **characteristic equation** of the recurrence relation $h_n - a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_k h_{n-k} = 0, \quad (a_k \neq 0, n \geq k)$.

Let q be a nonzero number. Then $h_n = q^n$ is a solution of the linear homogeneous recurrence relation

$h_n - a_1 h_{n-1} - a_2 h_{n-2} = 0, \quad (a_2 \neq 0, n \geq 2) \quad (*)$

with constant coefficients if and only if q is a root of the polynomial equation

$x^2 - a_1 x - a_2 = 0. \quad (**)$

If the characteristic equation $x^2 - a_1 x - a_2 = 0$ of the recurrence relation $h_n = a_1 h_{n-1} + a_2 h_{n-2}$ $(*)$ has two distinct non-zero roots q_1 and q_2 then $h_n = c_1 q_1^n + c_2 q_2^n$ is the general solution of the recurrence relation $h_n = a_1 h_{n-1} + a_2 h_{n-2}$.

If the characteristic equation $x^2 - a_1 x - a_2 = 0$ of the recurrence relation $h_n = a_1 h_{n-1} + a_2 h_{n-2}$ $(*)$ has a single non-zero root q_1, then the general solution of $(*)$ is $h_n = c_1 q_1^n + c_2 n q_1^n$.

For example:

(i) Solve the recurrence relation $h_n = 4h_{n-1} + 5h_{n-2}, \quad h_0 = 2, h_1 = 6$.
Solution: The characteristic equation associated with the given recurrence relation is $x^2 - 4x - 5 = 0$
The roots of the equation are $q_1 = 5, q_2 = -1$
The two roots are distinct.
$\therefore$ the general solution of $h_n = 4h_{n-1} + 5h_{n-2}$ is $h_n = c_1 5^n + c_2(-1)^n$
Now we want to find c_1, c_2 such that $h_0 = 2, h_1 = 6$.
$2 = h_0 = c_1 + c_2 \Longrightarrow c_1 + c_2 = 2 \quad (1)$
$6 = h_1 = c_1 * q_1 + c_2 * q_2 = 5c_1 - c_2 \Longrightarrow 5c_1 - c_2 = 6 \quad (2)$
Adding the two equations, $6c_1 = 8 \Longrightarrow c_1 = \frac{4}{3}$.
$c_2 = 2 - \frac{4}{3} = \frac{2}{3}$.
$\therefore h_n = \frac{4}{3}5^n + \frac{2}{3}(-1)^n$.

(ii) Find the general solution of the recurrence relation $h_n - 4h_{n-1} + 4h_{n-2} = 0, (n \geq 2),\ h_0 = 1, h_1 = 6$ using the characteristic equation.

Solution: The characteristic equation of this recurrence relation is $x^2 - 4x + 4 = 0$
The roots of the equation are $q_1 = 2, q_2 = 2$.
So, the roots are repeated.
$\therefore$ the general solution of $h_n - 4h_{n-1} + 4h_{n-2} = 0$ is $h_n = c_1\ 2^n + c_2\ n\ 2^n$.
We find c_1 and c_2 such that $h_0 = 1, h_1 = 6$.

$$\begin{aligned} c_1 2^0 + c_2 * 0 * 2^0 &= h_0 = 1 \\ c_1 &= 1 \qquad (*) \\ c_1 2^1 + c_2 * 1 * 2^1 &= h_1 = 6 \\ 1 * 2 + 2c_2 &= 6 \\ c_2 &= 2 \qquad (**) \end{aligned}$$

Hence the solution of $h_n - 4h_{n-1} + 4h_{n-2} = 0, h_0 = 1, h_1 = 6$ is
$h_n = 1 * 2^n + 2 * n2^n = 2^n + n2^{n+1} \quad n \geq 0$

4.6

4.6.2 PRACTICAL 2.6

(A) Objective Questions

Choose correct alternative in each of the following:

(1) For the sequence $1, 7, 25, 79, 241, 727, \ldots,$ simple formula for (a_n) is

(a) $3^{n+2} - 2$ (b) $3^n - 2$ (c) $-3^n + 4$ (d) $n^2 - 2$

(2) For the sequence $a_n = 6(1/3)^n$, a_4 is

(a) $\frac{2}{25}$ (b) $\frac{2}{27}$ (c) $\frac{2}{19}$ (d) $\frac{2}{13}$

(3) The recurrence system with initial condition $a_0 = 0$ and recurrence relation $a_n = a_{n-1} + 2n - 1$ is linear of

(a) degree two and non-homogeneous.
(b) degree one and non-homogeneous.
(c) degree one and homogeneous.
(d) None of these

(4) Fibonacci Numbers with $f_0 = 1, f_1 = 1$ fibonacci recurrence relation $f_n = f_{n-1} + f_{n-2}$ is linear of

(a) degree one and homogeneous
(b) degree two and non-homogeneous
(c) degree two and homogeneous
(d) None of these

(5) The recurrence relation for the number of ways to arrange n distinct objects in a row is

(a) $h_n = h_{n-1} + n, \quad h_1 = 1$
(b) $h_n = nh_{n-1}, \quad h_1 = 1$
(c) $h_n = nh_{n-1}, \quad h_1 = 0$
(d) None of these

(6) An elf has a staircase of n stairs to climb. Each step it takes can cover either one stair or two stairs . A recurrence relation for h_n, the number of different ways for the elf to ascend the $n-$stairs staircase is given by

(a) $h_n = nh_{n-1}+h_{n-2}$, $h_1 = 1, h_2 = 2, n \geq 3$.
(b) $h_n = h_{n-1} + h_{n-2}$, $h_1 = 1, h_2 = 2, n \geq 3$.
(c) $h_n = h_{n-1}+nh_{n-2}$, $h_1 = 1, h_2 = 2, n \geq 3$.
(d) None of these

(7) Consider the recurrence relation $a_n = -8a_{n-1} - 15a_{n-2}$ with initial conditions $a_0 = 0$ and $a_1 = 2$. Which of the following is an explicit solution to this recurrence relation?

(a) $a_n = (-3)^n + (5)^n$
(b) $a_n = n(-3)^n + n(5)^n$
(c) $a_n = n(-3)^n + (5)^n$
(d) None of these

(8) Consider the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 0$ and $a_1 = 2$. Which of the following is an explicit solution to this recurrence relation, provided the constants A and B are chosen correctly?

(a) $a_n = A3^n + B3^n$
(b) $a_n = A3^n + B(-3)^n$
(c) $a_n = A3^n + nB3^n$
(d) $a_n = A(-3)^n + B(-3)^n$

(9) Consider the recurrence relation $a_n = 2a_{n-1}$ with initial conditions $n \geq 1$ and $a_0 = 3$. Which of the following is an explicit solution to this recurrence relation?

(a) $a_n = 3.2^n$ (b) $a_n = 2.3^n$ (c) $a_n = 3.n^2$ (d) None of these

(10) Let the characteristic equation $x^2 - a_1x - a_2 = 0$ of the recurrence relation $h_n = a_1h_{n-1} + a_2h_{n-2}$ has two roots q_1 and q_2. If $h_n = c_1q_1^n + c_2q_2^n$ is the general solution of the recurrence relation of $h_n = a_1h_{n-1} + a_2h_{n-2}$ then q_1 and q_2 are

(a) Equal and non-zero
(b) Distinct
(c) non-zero
(d) None of these

(11) Let the characteristic equation $x^2 - a_1x - a_2 = 0$ of the recurrence relation $h_n = a_1h_{n-1} + a_2h_{n-2}$ has two roots q_1 and q_2. If $h_n = c_1q_1^n + c_2nq_2^n$ is the general solution of the recurrence relation of $h_n = a_1h_{n-1} + a_2h_{n-2}$ then q_1 and q_2 are

(a) Equal and non-zero
(b) Distinct and non-zero
(c) Equal
(d) None of these

(12) The characteristic polynomial corresponding to the recurrence $h_n = -25h_{n-1} + 54h_{n-2}$ is

(a) $25x^2 - 54x + 1$ (b) $-25x^2 + 54x$ (c) $x^2 + 25x - 54$ (d) $x^2 - 25x + 54$

(13) The characteristic polynomial corresponding to the recurrence $h_n = 2h_{n-1} - h_{n-2} + 13h_{n-3}$ is

(a) $2x^2 - x + 13$ (b) $-2x^2 + x - 13$ (c) $x^3 - 2x^2 + x - 13$ (d) $x^3 + 2x^2 - x + 13$

(14) A recursive linear homogeneous system of order k has

(a) exactly k characteristic roots.
(b) exactly $k - 1$ characteristic roots.
(c) exactly k distinct characteristic roots.
(d) may not have any characteristic roots.

(B) Descriptive Questions

(1) Find the recurrence relation for the number of ways to arrange n distinct objects in a row.

(2) An elf has a staircase of n stairs to climb. Each step it takes can cover either one stair or two stairs . Find a recurrence relation for h_n; the number of different ways for the elf to ascend the $n-$stairs staircase.

4.6

(3) Find the recurrence relation and give initial conditions for the number of binary strings of length n, that do not have two consecutive $0'$s.

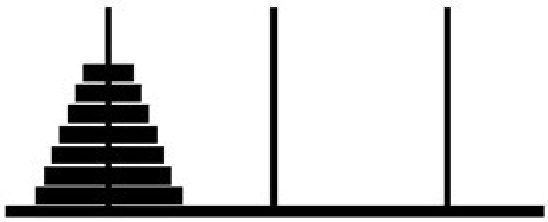
(4) Find a recurrence relation for h_n, the number of $n-$digit ternary sequences without any occurrence of the subsequence $'012'$.

(5) Suppose we draw n straight lines on a piece of paper so that every pair of lines intersect (but no three lines intersect at a common point). Into how many regions do these n lines divide the plane.

(6) Words of length n using only three letters a, b, c are to be transmitted over a communication channel subject to the condition that no word in which two a's appear consecutively is to be transmitted. Give a recurrence relation for the number of words of length n allowed by the communication channel and solve it.

(7) A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation and solve it for the number of pairs of rabbits on the island after n months, assuming that no rabbits ever die.

(8) The Tower of Hanoi consists of three pegs mounted on a board together with disks of different sizes. Initially these disks are placed on the first peg in order of size, with the largest at the bottom. The rules of the puzzle allow disks to be moved one at a time from one peg to another as long as a disk is never placed on top of a smaller disk.The goal of the puzzle is to have all the disks on the second peg in order of size, with the largest at the bottom. Let H_n denote the minimum number of moves needed to solve the Tower of Hanoi problem with n disks. Set up a recurrence relation for the sequence $\{H_n\}$ and solve it by back-tracking.



(9) Solve the following recurrence relations using iteration method.

(i) $a_n = -2a_{n-1}; n \geq 2, a_1 = 3$

(ii) $a_n = 3a_{n-1} + 7; n \geq 2, a_1 = 5$

(iii) $a_n = a_{n-1} + 3, a_1 = 2.$

(10) Solve the following linear homogeneous recurrence relations by using characteristic equation.

(i) $h_n = 5h_{n-1} - 6h_{n-2}; h_0 = 1, h_1 = 0$

(ii) $h_n - 4h_{n-1} + 4h_{n-2} = 0; n \geq 2, h_0 = 1, h_1 = 6$

(iii) $h_n - 6h_{n-1} + 9h_{n-2} = 0; n \geq 2, h_0 = 1, h_1 = 6$

(iv) $h_n = 2h_{n-1} + h_{n-2} - 2h_{n-3}; n \geq 3, h_0 = 1, h_1 = 2, h_2 = 0$

(v) $h_n = h_{n-1} + h_{n-2}; n \geq 2, h_0 = 0, h_1 = 1$

(vi) $h_n = 6h_{n-1} - 11h_{n-2} + 6h_{n-3}, h_0 = 2, h_1 = 5, h_2 = 15$

(11) Solve the following linear non-homogeneous recurrence relations.

(i) $h_n = 3h_{n-1} - 4n; h_0 = 2$ (ii) $h_n = 3h_{n-1} + 2n; h_0 = 3$

(12) A bank pays 8% interest each year on money in the savings account. Find a recurrence relation for the amounts of money a person would have after n years if it follows the investment strategy of (a) investing 1000 and leaving it in the bank for n years. (b) investing 1000 at the end of each year.

(13) A child has n rupees. Each day he buys either milk for Re. $1/-$ or Orange juice for Rs. $2/-$ or Pineapple juice for Rs.$2/-$. If h_n denotes the number of ways of spending all the money, find the recurrence relation for this sequence. In how many ways can he spend Rs. 7?

4.7 Practical 2.7: Miscellaneous theory questions

4.7.1 Miscellaneous theory questions from UNIT I

(1) Let m be a natural number. Show that the following statement is true for every natural number n: If there is an injective function from N_n to N_m, then $n \leq m$.

(2) Let A be a nonempty set, let $n \in N$. Show that the following statements are equivalent:

(a) There is a surjective function $f : \mathbb{N}_n \longrightarrow A$.

(b) There is an injective function $g : A \longrightarrow \mathbb{N}_n$.

(c) A is finite and $|A| \leq n$.

(3) If X, Y are finite sets and there is an injective function $f : X \to Y$ then show that $|X| \leq |Y|$.

(4) If the set S is such that there is a bijection $b : \mathbb{N} \to S$ then Show that S is infinite.

(5) Let A be a nonempty set. Show that the following are equivalent:

(a) There is a surjective function $f : \mathbb{N} \longrightarrow A$.

(b) There is an injective function $g : A \longrightarrow \mathbb{N}$.

(c) A is countable (that is finite or countably infinite).

(6) Show that the set $\mathbb{Z}$ of all integers is countably infinite.

(7) Show that every infinite subset of $\mathbb{N}$ is countably infinite.

(8) If A, B are countable sets then $A \times B$ is also countable.

(9) If A_1, A_2 are countable then $A_1 \cup A_2$ is also countable.

(10) State and prove Addition Principle and Multiplication Principle of Counting.

4.7

(11) Let X and Y be finite non-empty sets, and S be a subset of $X \times Y$. Let $R_x(S)$ be the set of pairs in S whose first coordinate is x and $C_y(S)$ be the set of pairs in S whose second coordinate is y. Then prove the following.

(a) The size of S is given by $|S| = \sum_x |R_x(S)| = \sum_y |C_y(S)|$.

(b) If $|R_x(S)|$ is a constant r, independent of x and $|C_y(S)|$ is a constant c, independent of y, then $r|X| = c|Y|$.

(12) Show the number of ways of putting n balls of distinct colours into k distinct boxes with each box containing at least one ball is $k!S(n,k)$, where $S(n,k)$ is the Stirling number of second type.

(13) Let n and k be positive integers. Show that the number of surjective functions from an $n-$set to a $k-$set is equal to $k!S(n,k)$, where $S(n,k)$ is the Stirling number of second type.

(14) Let n and m be positive integers and k be an integer such that $1 \leq k \leq m$. Prove that the number of functions from a n-set to a m-set is $m^n = \sum_{k=1}^{m} \binom{m}{k} k!S(n,k)$.

(15) Define Stirling number $S(n,k)$ of second kind. Let n and k be positive integers with $n \geq k$. Show that $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.

(16) For all $n \geq 2$. Show that $S(n,2) = 2^{n-1} - 1$.

4.7.2 Miscellaneous theory questions from UNIT II

(1) State and prove Binomial Theorem.

(2) Let n be a non-negative integer. Show that

$$(x+y)^n = \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{n} y^n.$$

(3) Define multinomial coefficient. Let S be an n-set and suppose the n objects in S are to be put in r distinct boxes $B_1, B_2, \ldots, B_r$ such that the i^{th} box B_i contains n_i objects with $n_1 + n_2 + \cdots + n_r = n$. Show that the number of ways of doing this is equal to

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1!n_2!\ldots n_r!}.$$

(4) State and prove Multinomial theorem.

(5) Let n be a non-negative integer. Show that

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{n_1+n_2+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} x_1^{n_1} x_2^{n_2} \ldots x_r^{n_r}$$

where the summation extends over all nonnegative integers $n_1 + n_2 + \cdots + n_r = n$.

(6) Let n and k be positive integers. Show that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$.

(7) Let S be a multi-set consisting of k distinct objects, each with infinite multiplicity. Show that the total number of r-permutations of S is k^r.

(8) Let S be a multi-set with k distinct objects with finite repetition numbers $n_1, n_2, \ldots, n_k$ respectively. If the size of S is $n = n_1 + n_2 + \cdots + n_k$, then show that the number of permutations of S equals $\frac{n!}{n_1!n_2!\ldots n_k!}$.

(9) Prove that the number of circular r-permutations of a set of n elements is given by $\frac{P(n,r)}{r} = \frac{n!}{r*(n-r)!}$.

(10) For $0 \le r \le n$, Prove that $P(n,r) = r! * \begin{pmatrix} n \\ r \end{pmatrix}$. Hence prove that

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r! * (n-r)!}$$

(11) Let S be a multi-set with objects of k different types each with an infinite repetition number (multiplicity). Show that the number of r-combinations of S equals

$$\begin{pmatrix} r+k-1 \\ r \end{pmatrix} = \begin{pmatrix} r+k-1 \\ k-1 \end{pmatrix}.$$

(12) Show that the number of ways of putting r identical objects into k distinct boxes with each box containing at least one object is $\begin{pmatrix} r-1 \\ k-1 \end{pmatrix}$.

(13) State and prove Inclusion-Exclusion principle.

(14) Show that the number of derangements D_n of $\{1, 2, \ldots, n\}$ is given by

$$D_n = \left\{ \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} \right\}.$$

(15) Let $n \ge 2$ be an integer whose prime factorization is $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ where $\alpha_i \ge 1$, $\forall i, 1 \le i \le r$, prove that

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2 1}\right) \ldots \left(1 - \frac{1}{p_r}\right).$$

4.7.3 Miscellaneous theory questions from UNIT III

(1) In S_n, if the pair of cycles $\alpha = (a_1\ a_2\ \ldots\ a_m)$ and $\beta = (b_1\ b_2\ \ldots\ b_t)$ have no entries in common, then show that $\alpha\beta = \beta\alpha$.

(2) Define signature of a permutation. If σ is any permutation in S_n, then show that the sign of σ is ± 1.

(3) If a permutation $\sigma \in S_n$ can be written as a product of r transpositions and also a product of r' transpositions, then show that either r and r' are both even or r and r' are both odd. (i.e. r and r' have the same parity).

(4) Prove that, for any integer $n \geq 2$, exactly half of the permutations in S_n are odd and half are even.

(5) If $\alpha, \beta \in S_n$, then show that $\text{sgn}(\alpha\beta) = \text{sgn}(\alpha).\text{sgn}(\beta)$.

(6) Prove that for $n > 1$, A_n has order $\frac{n!}{2}$.

(7) Define linear homogeneous recurrence relation. Let q be a nonzero number. Show that $h_n = q^n$ is a solution of the linear homogeneous recurrence relation

$$h_n - a_1h_{n-1} - a_2h_{n-2} - \cdots - a_kh_{n-k} = 0, (a_k \neq 0, n \geq k) - (1)$$

with constant coefficients if and only if q is a root of the polynomial equation $x^k - a_1x^{k-1} - a_2x^{k-2} - \cdots - a_k = 0 - (2)$. Hence prove that if the polynomial equation has k distinct roots $q_1, q_2, \ldots, q_k$ then $h_n = c_1q_1^n + c_2q_2^n + \cdots + c_kq_k^n$ is the general solution of (1).

(8) Show that if the characteristic equation $x^2 - a_1x - a_2 = 0$ of the recurrence relation $h_n = a_1h_{n-1} + a_2h_{n-2}$ has two distinct non-zero roots q_1 and q_2 then $h_n = c_1q_1^n + c_2q_2^n$ is the general solution of the recurrence relation of $h_n = a_1h_{n-1} + a_2h_{n-2}$.

(9) Show that if the characteristic equation $x^2 - a_1x - a_2 = 0$ of the recurrence relation $h_n = a_1h_{n-1} + a_2h_{n-2}$ has a single non-zero roots q_1 then $h_n = c_1q_1^n + c_2nq_2^n$ is the general solution of the recurrence relation of $h_n = a_1h_{n-1} + a_2h_{n-2}$.

(10) Let $h_0, h_1, \ldots, h_n$ is a sequence of real numbers. When do we say that this sequence satisfies a linear recurrence relation of order k? Also show that there exist constants $c_1, c_2, \ldots, c_k$ such that $h_n = c_1q_1^n + c_2q_2^n + \cdots + c_kq_k^n$ satisfies $h_0 = b_0, h_1 = b_1, \ldots, h_{k-1} = b_{k-1}$ where $q_1, q_2, \ldots, q_k$ are distinct real numbers and $b_0, b_1, \ldots, b_{k-1}$ are any k real numbers.

xxxxxxxxxxxx

Index